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Lesson H

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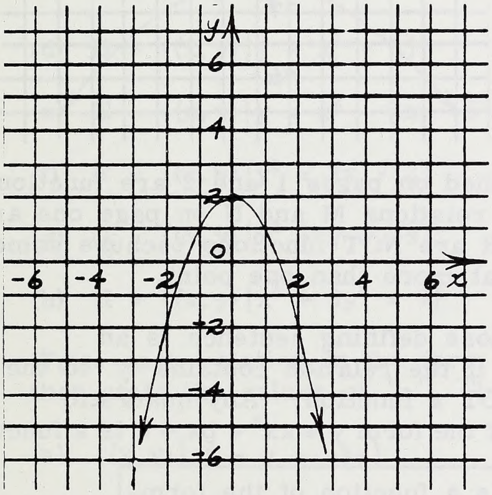
QUADRATIC FUNCTIONS AND RELATIONS

Topic One: Quadratic Functions

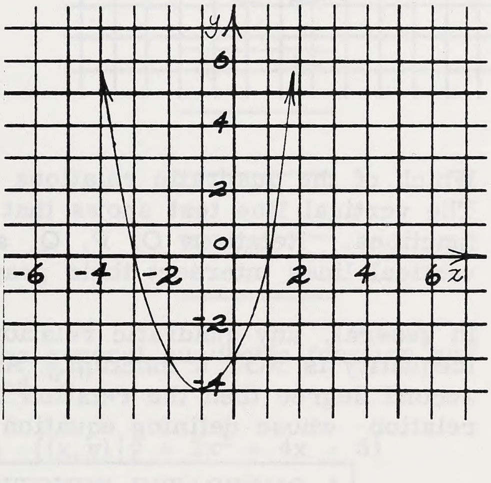
A. Definition of a Quadratic Function

A relation is composed of a set of ordered pairs. (See page 24, lesson G.) Where the defining sentence of a relation is a quadratic equation or inequality in two variables, the relation is of the second degree and is called a QUADRATIC RELATION. The relations M, N, O, P, Q, and R which are graphed on pages 1 and 2 are examples of quadratic relations. Notice the second degree equations and inequalities in two variables.

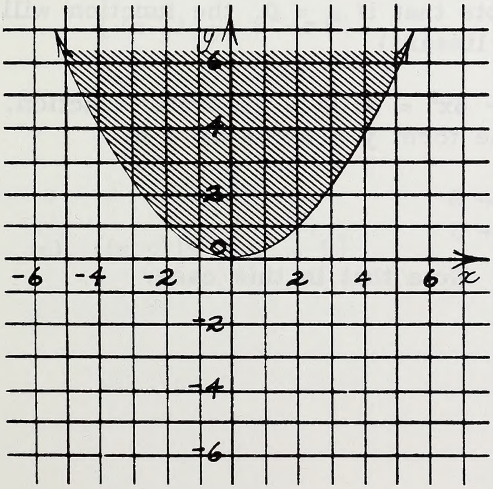
$M = \{(x, y) \mid y = -x^2 + 2\}.$



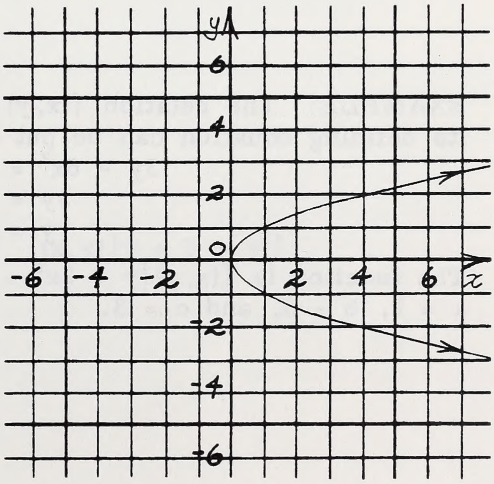
$N = \{(x, y) \mid y = x^2 + 2x - 3\}.$



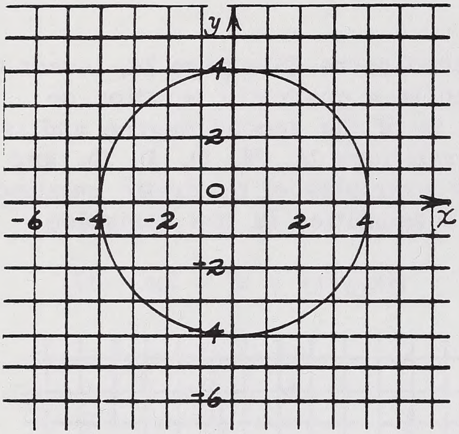
$O = \{(x, y) \mid y \geqq \frac{1}{4}x^2\}.$



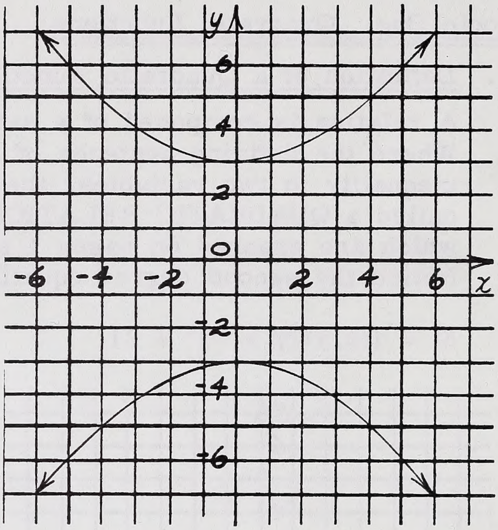
$P = \{(x, y) \mid y^2 = x\}.$



$Q = \{(x, y) \mid x^2 + y^2 = 16\}$.



$R = \{(x, y) \mid y^2 - x^2 = 9\}$.



Which of the quadratic relations graphed on pages 1 and 2 are functions? The vertical line test shows that the relations M and N on page one are functions. Relations O, P, Q, and R are NOT functions because some vertical lines intersect their graphs at more than one point.

In general, any quadratic relation whose defining sentence is an inequality is NOT a function. Also, if the relation contains y to the second degree then the relation is NOT a function. Any quadratic relation whose defining equation is of the form $y = ax^2 + bx + c$ is a function.

> A QUADRATIC FUNCTION is a function of the form
> $q = \{(x, y) \mid y = ax^2 + bx + c\}$
> where $a, b, c \in R$ and $a \neq 0$.

(Note that if $a = 0$, the function will be linear.)

EXAMPLE: The relation $\{(x, y) \mid 3y - 6x^2 = 9\}$ is a quadratic function. Its defining equation can be put in the form $y = ax^2 + bx + c$.

$$\begin{aligned} 3y - 6x^2 &= 9 \\ 3y &= 6x^2 + 9 \\ y &= 2x^2 + 3 \end{aligned}$$

The function is $\{(x, y) \mid y = 2x^2 + 3\}$. Note that in this case, $a = 2$, $b = 0$, and $c = 3$.

Self-correcting Exercise #1

Answers may be found on page 59 of this lesson.

1. Decide whether or not each of the following relations is a quadratic function.

Yes or No?

(a) $A = \{(x, y) | x + 3y = 5\}$ ________

(b) $B = \{(x, y) | x^2 = 5 - y^2\}$ ________

(c) $C = \{(x, y) | y - x = x^2\}$ ________

(d) $D = \{(x, y) | y > x^2 - 5\}$ ________

(e) $E = \{(x, y) | y = x^2 - 3x - 5\}$ ________

(f) $F = \{(x, y) | y^2 = 3x - 2\}$ ________

(g) $G = \{(x, y) | x^2 - y = 5x\}$ ________

(h) $H = \{(x, y) | x^2 + 3y^2 = 9\}$ ________

2. Write each relation in the form of the general quadratic function and then state the values of "a", "b", and "c".

(a) $\{(x, y) | x^2 - y = -3x\}$

$x^2 - y = -3x$

$-y = -x^2 - 3x$

$y = x^2 + 3x$

$\{(x, y) | y = x^2 + 3x\}$

$a = 1,\ b = 3,\ c = 0$

(b) $\{(x, y) | y + 2x^2 = 4x - 8\}$

(c) $\{(x, y) | y - x^2 = 0\}$

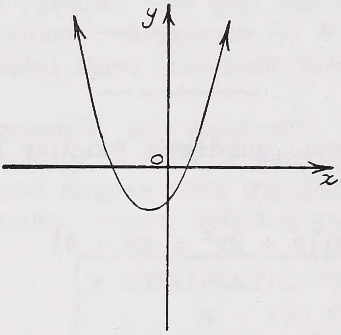
(d) $\{(x, y) | y + 3 = x^2\}$

(e) $\{(x, y) \mid 2x^2 - 5x = y + 9\}$

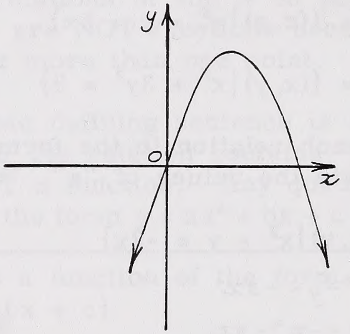
(f) $\{(x, y) \mid y - 4x = 2x^2 + 1\}$

B. The Graph of a Quadratic Function

The graph of the quadratic function $\{(x,y) \mid y = ax^2 + bx + c\}$ is a PARABOLA which opens upward or downward in the coordinate plane.



Parabola which opens upward

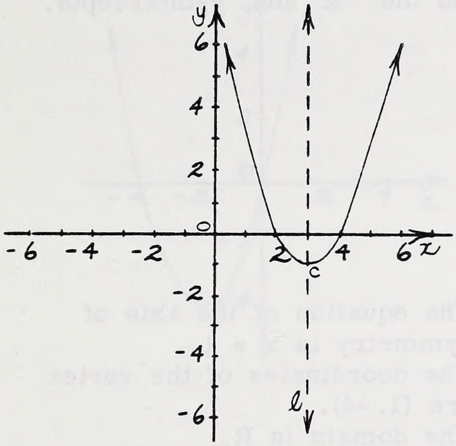


Parabola which opens downward

A parabola is symmetric with respect to a line called its AXIS OF SYMMETRY. If a parabola is folded along its axis of symmetry, the two halves of the parabola will coincide.

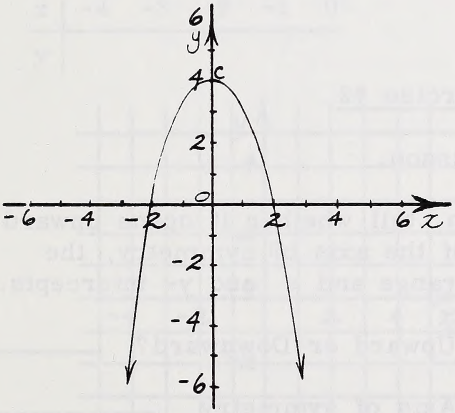
The axis of symmetry of a parabola intersects the parabola at one point called the VERTEX. If the parabola opens upward, the vertex is the lowest or minimum point on the graph. All other points on the graph have y-coordinates which are greater than the y-coordinate of the vertex. If the parabola opens downward, the vertex is the highest or maximum point on the graph. All other points on the graph have y-coordinates which are less than the y-coordinate of the vertex.

EXAMPLE 1:



This parabola opens upward. It is symmetric with respect to the line ℓ whose defining equation is x = 3. Point C is the vertex of the parabola. Its coordinates are (3, -1). The vertex is the minimum point on the parabola. The range of the function is $\{y | y \geqq -1\}$.

EXAMPLE 2:



This parabola opens downward. It is symmetric with respect to the y-axis whose defining equation is x = 0. Point C is the vertex of the parabola. Its coordinates are (0, 4). The vertex is the maximum point on the parabola. The range of the function is $\{y | y \leqq 4\}$.

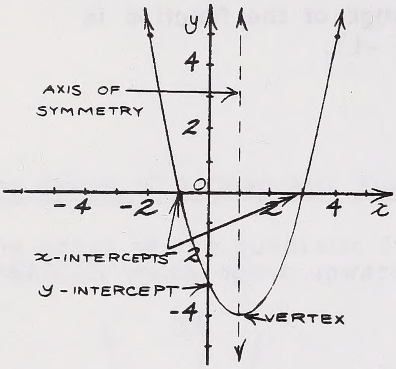
The parabolic graph of a quadratic function can be drawn by locating a few points whose coordinates satisfy the defining equation and joining these points by a smooth curve.

EXAMPLE: Graph the quadratic function $\{(x, y) \mid y = x^2 - 2x - 3\}$. Then, give the equation of the axis of symmetry, the coordinates of the vertex, the domain and range of the function and the x and y-intercepts.

Solution

$y = x^2 - 2x - 3$

x	-2	-1	0	1	2	3	4
y	5	0	-3	-4	-3	0	5



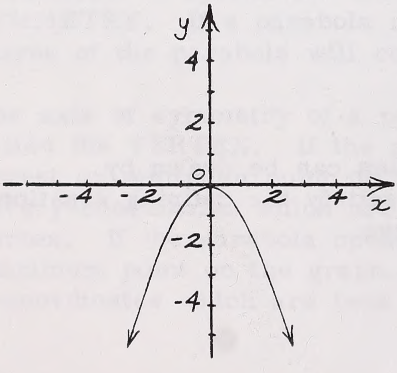
(1) The equation of the axis of symmetry is $x = 1$.
(2) The coordinates of the vertex are $(1, -4)$.
(3) The domain is R.
(4) The range is $\{y \mid y \geqq -4\}$.
(5) The x-intercepts are -1 and 3.
(6) The y-intercept is -3.

Self-correcting Exercise #2

Answers may be found on page 59 of this lesson.

1. For the graph of each quadratic function, tell whether it opens upward or downward. Then give the equation of the axis of symmetry, the coordinates of the vertex, the domain, range and x and y- intercepts.

(a)



(i) Upward or Downward? __________

(ii) Axis of symmetry __________

(iii) Vertex __________

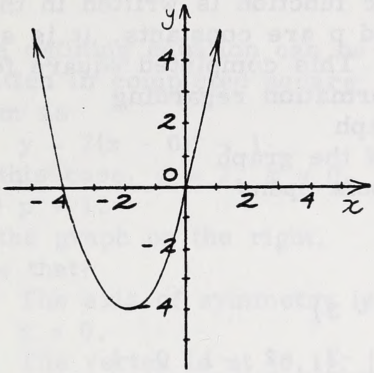
(iv) Domain __________

(v) Range __________

(vi) x-intercept __________

(vii) y-intercept __________

(b)



(i) Upward or Downward? ________

(ii) Axis of symmetry ________

(iii) Vertex ________

(iv) Domain ________

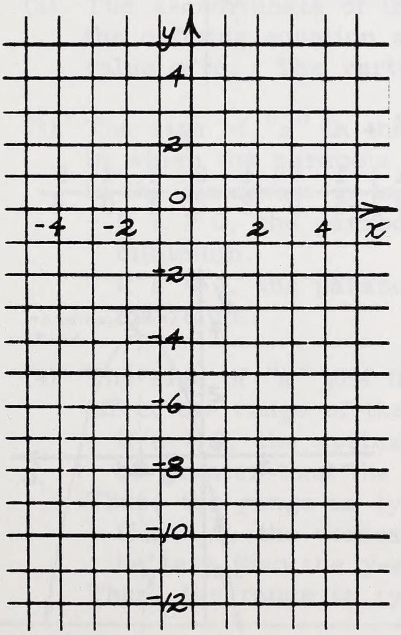
(v) Range ________

(vi) x-intercepts ________

(vii) y-intercepts ________

2. Graph the quadratic function $\{(x, y) \mid y = -2x^2 - 8x - 11\}$. Then state the equation of the axis of symmetry and the coordinates of the vertex.

x	-4	-3	-2	-1	0
y					



C. Quadratic Functions of the Form $\{(x, y) \mid y = a(x - k)^2 + p\}$

When the defining equation of a quadratic function is written in the form $y = a(x - k)^2 + p$, where a, k, and p are constants, it is said to be in COMPLETED SQUARE FORM. This completed square form of the equation gives us some valuable information regarding

(1) the axis of symmetry of the graph
(2) the coordinates of the vertex of the graph
(3) the direction in which the parabola opens
(4) the range of the function.

Study the following examples.

EXAMPLE 1: $\{(x, y) \mid y = 2(x + 1)^2 - 3\}$

x	-3	-2	-1	0	1
y	5	-1	-3	-1	5

The defining equation is

$$y = 2(x + 1)^2 - 3.$$

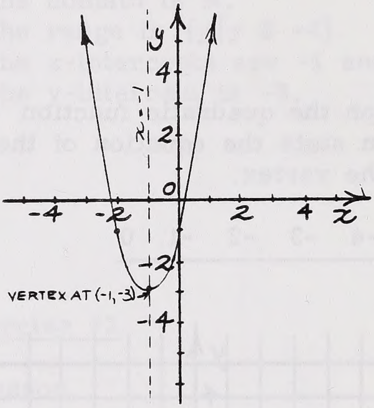
This equation can be written in completed square form as

$$y = 2[x - (-1)]^2 + (-3).$$

In this case, $a = 2$, $k = -1$, and $p = -3$.

In the graph on the right, note that:

(1) The axis of symmetry is $x = -1$.
(2) The vertex is at $(-1, -3)$.
(3) The graph opens upward and the vertex is a minimum.
(4) The range of the function is $\{y \mid y \geqq -3\}$.



EXAMPLE 2: $\{(x, y) \mid y = -(x - 2)^2 + 4\}$

x	-1	0	1	2	3	4	5
y	-5	0	3	4	3	0	-5

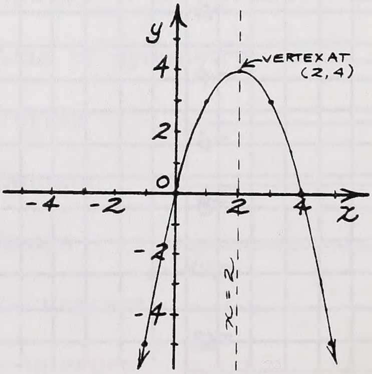
The defining equation can be written in completed square form as

$$y = -1[x - (2)]^2 + (4).$$

In this case, $a = -1$, $k = 2$, and $p = 4$.

In the graph on the right, note that:

(1) The axis of symmetry is $x = 2$.
(2) The vertex is at $(2, 4)$.
(3) The graph opens downward and the vertex is a maximum.
(4) The range of the function is $\{y \mid y \leqq 4\}$.



EXAMPLE 3: $\{(x, y) \mid y = 2x^2 + 1\}$

x	-3	-2	-1	0	1	2	3
y	19	9	3	1	3	9	19

The defining equation can be written in completed square form as

$$y = 2(x - 0)^2 + 1.$$

In this case, $a = 2$, $k = 0$, and $p = 1$.

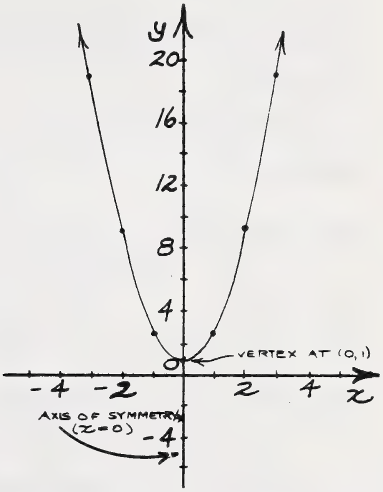
In the graph on the right, note that:

(1) The axis of symmetry is $x = 0$.

(2) The vertex is at $(0, 1)$.

(3) The graph opens upward and the vertex is a minimum.

(4) The range of the function is $\{y \mid y \geqq 1\}$.



In general, for the quadratic function $q = \{(x, y) \mid y = a(x - k)^2 + p\}$:

(1) The axis of symmetry of the graph of the function depends upon the value of k in the defining equation. The axis of symmetry is the line $x = k$.

(2) The x-coordinate of the vertex depends upon the value of k in the defining equation and the y-coordinate depends upon the value of p. The vertex occurs at the point (k, p).

(3) The sign of "a" in the defining equation affects the direction in which the parabola opens and determines whether the vertex is the maximum or minimum point on the graph.
- If $a > 0$, the parabola opens upward and the vertex is a minimum.
- If $a < 0$, the parabola opens downward and the vertex is a maximum.

(4) The sign of "a" and the value of "p" in the defining equation affect the range of the function.
- If $a > 0$, the vertex is a minimum and all values of y must be greater than the y-coordinate at the vertex.

Thus, the range is $\{y \mid y \geqq p\}$.
- If $a < 0$, the vertex is a maximum and all values of y must be less than the y-coordinate at the vertex.

Thus, the range is $\{y \mid y \leqq p\}$.

EXAMPLE 1

$q_1 = \{(x, y) \mid y = -3x^2\}$

The defining equation can be written in completed square form as:

$$y = -3(x - 0)^2 + 0$$

In this case, $a = -3$, $k = 0$, and $p = 0$.

(1) The axis of symmetry is $x = k$, or in this case, $x = 0$.

(2) Coordinates of vertex are $(k, p) = (0, 0)$.

(3) Since "a" is negative ($a = -3$), the parabola opens downward and the vertex is a maximum.

(4) Since "a" is negative and $p = 0$, the range is

$$\{y \mid y \leqq 0\}.$$

EXAMPLE 2

$q_2 = \{(x, y) \mid y = (x + 1)^2 - 2\}$

The defining equation can be written in completed square form as:

$$y = 1[x - (-1)]^2 + (-2)$$

In this case, $a = 1$, $k = -1$, and $p = -2$.

(1) The axis of symmetry is $x = k$, or in this case, $x = -1$.

(2) Coordinates of vertex are $(k, p) = (-1, -2)$.

(3) Since "a" is positive ($a = 1$), the parabola opens upward and the vertex is a minimum.

(4) Since "a" is positive and $p = -2$, the range is

$$\{y \mid y \geqq -2\}.$$

Self-correcting Exercise #3

Answers may be found on page 60 of this lesson.

1. Write each of the following equations in the form $y = a(x - k)^2 + p$ and then state the values of a, k, and p.

(a) $y = 3x^2 - 7$

(b) $y = 5(x - 1)^2$

(c) $y = -2(x + 1)^2 - 4$

(d) $y = -(x + 3)^2$

(e) $y = 4x^2$

(f) $y = x^2 + 1$

2. For each of the following functions, give the equation of the axis of symmetry and state the coordinates of the vertex.

	Axis of Symmetry	Vertex
(a) $\{(x, y) \mid y = 4(x - 1)^2 + 3\}$	________	________
(b) $\{(x, y) \mid y = -(x + 3)^2 - 7\}$	________	________
(c) $\{(x, y) \mid y = -5x^2\}$	________	________
(d) $\{(x, y) \mid y = 2(x - 1)^2\}$	________	________
(e) $\{(x, y) \mid y = (x + 4)^2 - 2\}$	________	________
(f) $\{(x, y) \mid y = 3x^2 - 5\}$	________	________
(g) $\{(x, y) \mid y = 3(x + 7)^2 - 1\}$	________	________
(h) $\{(x, y) \mid y = x^2 - 5\}$	________	________

3. For each of the following functions, decide if its graph opens upward or downward and if the vertex is a maximum or minimum.

	Up or Down?	Max. or Min.?
(a) $\{(x, y) \mid y = 2x^2\}$	________	________
(b) $\{(x, y) \mid y = -(x - 5)^2 + 2\}$	________	________
(c) $\{(x, y) \mid y = -3x^2 + 7\}$	________	________
(d) $\{(x, y) \mid y = (x + 1)^2\}$	________	________
(e) $\{(x, y) \mid y = -2(x - 3)^2 + 5\}$	________	________
(f) $\{(x, y) \mid y = 3(x - 1)^2 - 1\}$	________	________

4. For each of the following functions, decide if "a" is positive or negative, give the value of p, and state the range of the function.

(a) $\{(x, y) \mid y = -(x - 2)^2 + 3\}$

"a" is ________________.

Value of p is ________ .

Range is ________________.

(b) $\{(x, y) \mid y = 2(x - 5)^2\}$

"a" is ________________.

Value of p is ________.

Range is ________________.

(c) $\{(x, y) \mid y = x^2 + 7\}$

"a" is ______________.

Value of p is _______.

Range is ______________.

(d) $\{(x, y) \mid y = -3(x + 1)^2 - 4\}$

"a" is ______________.

Value of p is _______.

Range is ______________.

5. Fill in the blanks in the following chart.

DEFINING EQUATION	AXIS OF SYMMETRY	COORDINATES OF VERTEX	RANGE OF FUNCTION
(a) $y = -2(x - 3)^2 + 8$	$x = 3$	(3, 8)	$\{y \mid y \leqq 8\}$
(b) $y = (x + 3)^2 + 1$			
(c) $y = -(x - 4)^2$			
(d) $y = 3(x + 1)^2 - 2$			
(e) $y = 4x^2$			
(f) $y = -3x^2 + 5$			
(g) $y = 7(x + 6)^2 - 1$			
(h) $y = -3(x - 5)^2 + 3$			

D. Writing the Defining Equation of a Quadratic Function in Completed Square Form

The defining equation of a quadratic function is generally written in the form $y = ax^2 + bx + c$. How can we change such an equation to the completed square form $y = a(x - k)^2 + p$?

Equations which contain no first degree term in x can easily be changed to completed square form.

EXAMPLES:

$y = 2x^2$ can be written as $y = 2(x - 0)^2 + 0$.
In this case, $a = 2$, $k = 0$, and $p = 0$.

$y = -3x^2 - 7$ can be written as $y = -3(x - 0)^2 - 7$. In this case, $a = -3$, $k = 0$, and $p = -7$.

Equations which contain a first degree term in x can be put into completed square form by using the following procedure.

PROCEDURE	EXAMPLE
Step 1. Enclose the second and first degree terms in x within parentheses.	$y = -4x^2 - 5x + 3$ $y = (-4x^2 - 5x) + 3$
Step 2. Divide the two terms within parentheses by the numerical coefficient of the second degree term. To compensate for doing this, you must also multiply both terms by that same number.	$y = -4\left(\frac{-4x^2}{-4} + \frac{-5x}{-4}\right) + 3$ (Divide both terms by -4.) (Multiply both terms by -4.) $y = -4(x^2 + \frac{5}{4}x) + 3$
Step 3. Find the term which must be added to the binomial inside the parentheses in order to make it a perfect square trinomial. (This term can be found by taking one-half the numerical coefficient of the linear term and squaring the result.)	What term must be added to $\left(x^2 + \frac{5}{4}x\right)$ so it will be a perfect square trinomial? The numerical coefficient of the linear term is $\frac{5}{4}$. The square of $\frac{1}{2}$ this coefficient is: $\left(\frac{1}{2} \times \frac{5}{4}\right)^2 = \left(\frac{5}{8}\right)^2 = \frac{25}{64}$.
Step 4. Add the same amount to each side of the equation.	$y + (-4)\left(\frac{25}{64}\right) = -4\left(x^2 + \frac{5}{4}x + \frac{25}{64}\right) + 3$ $\left[\text{Add } (-4)\left(\frac{25}{64}\right)\right]$ $\left[\text{Add } (-4)\left(\frac{25}{64}\right)\right]$
Step 5. Simplify the equation until it is written in the form $y = a(x - k)^2 + p$.	$y - \frac{25}{16} = -4\left(x + \frac{5}{8}\right)^2 + 3$ $y = -4\left(x + \frac{5}{8}\right)^2 + 3 + \frac{25}{16}$ $y = -4\left(x + \frac{5}{8}\right)^2 + \frac{73}{16}$
Step 6. State the equation of the axis of symmetry and the coordinates of the vertex.	In this case, $a = -4$, $k = \frac{-5}{8}$, and $p = \frac{73}{16}$. The equation of the axis of symmetry is $x = \frac{-5}{8}$ and the coordinates of the vertex are $\left(\frac{-5}{8}, 4\frac{9}{16}\right)$.

Practice on Step 1 and Step 2. Complete the following table:

EQUATION	STEP 1	STEP 2
$y = 2x^2 - 10x - 10$	$y = (2x^2 - 10x) - 10$	$y = 2(x^2 - 5x) - 10$
$y = 3p^2 + 9p - 2$	$y = (3p^2 + 9p) - 2$	$y = 3(p^2 + 3p) - 2$
$y = -4x^2 + 12x + 8$	$y = (-4x^2 + 12x) + 8$	$y = -4(x^2 - 3x) + 8$
$y = 9x^2 - 6x + 4$		
$y = -3x^2 + 6x - 1$		
$y = 6x^2 + 7x - 3$		
$y = -x^2 - 2x + 9$		
$y = -5x^2 - 15x + 9$		
$y = 3x^2 - 9x - 4$		

EXAMPLE: Write $y = 10x^2 - 19x + 9$ in the completed square form.

Solution

$$y = 10x^2 - 19x + 9$$

$$y = (10x^2 - 19x) + 9$$

$$y = 10\left(x^2 - \frac{19}{10}x\right) + 9$$

$$y + 10\left(\frac{361}{400}\right) = 10\left(x^2 - \frac{19}{10}x + \frac{361}{400}\right) + 9$$

Note that: $\left(\frac{1}{2} \times \frac{-19}{10}\right)^2 = \left(\frac{-19}{20}\right)^2 = \frac{361}{400}$

$$y + \frac{361}{40} = 10\left(x - \frac{19}{20}\right)^2 + 9$$

$$y = 10\left(x - \frac{19}{20}\right)^2 + 9 - \frac{361}{40} \quad \text{where} \quad 9 - \frac{361}{40} = \frac{360}{40} - \frac{361}{40} = \frac{-1}{40}$$

$$y = 10\left(x - \frac{19}{20}\right)^2 - \frac{1}{40}$$

Practice on Step 3 and Step 4. See the pattern in the first three rows of the following table and then complete the remaining three rows.

EQUATION IN STEP 2 FORM	STEP 3	Value of "a"	STEP 4
$y = (x^2 + 4x) + 3$	$\left(\frac{4}{2}\right)^2 = (2)^2 = 4$	$a = 1$	$y + 1(4) = (x^2 + 4x + 4) + 3$
$y = 2(x^2 - 5x) - 10$	$\left(\frac{-5}{2}\right)^2 = \frac{25}{4}$	$a = 2$	$y + 2\left(\frac{25}{4}\right) = 2\left(x^2 - 5x + \frac{25}{4}\right) - 10$
$y = 3(p^2 + 3p) - 81$	$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$	$a = 3$	$y + 3\left(\frac{9}{4}\right) = 3\left(p^2 + 3p + \frac{9}{4}\right) - 81$
$y = (x^2 + 8x) + 3$	$\left(\frac{\quad}{2}\right)^2 =$	$a =$	$y + 1(\quad) = (x^2 + 8x + \underline{\quad}) + 3$
$y = 3(x^2 + 12x) - 1$	$\left(\frac{\quad}{2}\right)^2 =$	$a =$	$y + 3(\quad) = 3(x^2 + 12x + \underline{\quad}) - 1$
$y = -2(x^2 - 5x) + \frac{3}{4}$	$\left(\frac{\quad}{2}\right)^2 =$	$a =$	$y + (\quad)(\quad) = -2(x^2 - 5x + \underline{\quad}) + \frac{3}{4}$

Practice on Step 5 and Step 6. The following table contains some of the equations from the practice on Step 1 and Step 2. Complete this table.

STEP 5	STEP 6	
	Axis of Symmetry	Coordinates of Vertex
$y + \overset{1}{\cancel{2}}\left(\frac{25}{\underset{2}{\cancel{4}}}\right) = 2\left(x^2 - 5x + \frac{25}{4}\right) - 10$ $y + \frac{25}{2} = 2\left(x - \frac{5}{2}\right)^2 - 10$ (SQUARE ROOT OF 25/4) $y = 2\left(x - \frac{5}{2}\right)^2 - 10 - \frac{25}{2}$ $y = 2\left(x - \frac{5}{2}\right)^2 - \frac{45}{2}$	$x = \frac{5}{2}$	$\left(\frac{5}{2}, -22\frac{1}{2}\right)$
$y + 3\left(\frac{9}{4}\right) = 3\left(p^2 + 3p + \frac{9}{4}\right) - 2$ $y + \frac{27}{4} = 3\left(p + \frac{3}{2}\right)^2 - 2$ (SQUARE ROOT OF 9/4) $y = 3\left(p + \frac{3}{2}\right)^2 - 2 - \frac{27}{4}$ $y = 3\left(p + \frac{3}{2}\right)^2 - \frac{35}{4}$	$p = \frac{-3}{2}$	$\left(\frac{-3}{2}, -8\frac{3}{4}\right)$
$y + (-4)\left(\frac{9}{4}\right) = -4\left(x^2 - 3x + \frac{9}{4}\right) + 8$ $y - \underline{\quad} = -4(x - \quad)^2 + \underline{\quad}$ $y = -4(x - \quad)^2 + \underline{\quad} + \underline{\quad}$ $y = -4(x - \quad)^2 + \underline{\quad}$	$x = \underline{\qquad}$	(,)
$y + 9\left(\frac{1}{9}\right) = 9\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + 4$ $y + \qquad = 9(\qquad)^2 + \underline{\quad}$ $y =$ $y =$	$x = \underline{\qquad}$	(,)

Self-correcting Exercise #4

Answers may be found on page 61 of this lesson.

1. Fill in the blanks in the following table.

EQUATION	FORM $y = a(x - k)^2 + p$	a	k	p	AXIS OF SYMMETRY	COORDINATES OF VERTEX
(a) $y = x^2 - 3$	$y = 1(x - 0)^2 - 3$	1	0	-3	$x = 0$	(0, -3)
(b) $y = -5x^2$						
(c) $y = \frac{-1}{2}x^2 + \frac{1}{4}$						
(d) $y = 2x^2$						
(e) $y = 3x^2 + 1$						

2. Find the term which can be added to each binomial in order to make it a perfect square trinomial.

	Term to be Added
(a) $x^2 + 7x$	________________
(b) $x^2 - \frac{1}{2}x$	________________
(c) $x^2 + 4x$	________________
(d) $x^2 - 2x$	________________
(e) $x^2 - \frac{3}{4}x$	________________
(f) $x^2 + \frac{2}{3}x$	________________
(g) $x^2 - 9x$	________________
(h) $x^2 - x$	________________

3. Write each equation in completed square form. Then, state whether its graph opens upward or downward and give the equation of the axis of symmetry, the coordinates of the vertex and the range of the function.

(a) $y = x^2 + 2x + 3$

$y = (x^2 + 2x) + 3$

$y + 1 = (x^2 + 2x + 1) + 3$

$y = (x+1)^2 + 3 - 1$

$y = (x+1)^2 + 2$

Upward or downward? ______________

Axis of symmetry ______________

Vertex ______________

Range ______________

(b) $y = 2x^2 + 3x$

Upward or downward? ______________

Axis of symmetry ______________

Vertex ______________

Range ______________

(c) $y = -3x^2 + 15x - 5$

Upward or downward? ______________

Axis of symmetry ______________

Vertex ______________

Range ______________

(d) $y = -x^2 + 2x - 2$

Upward or downward? ______________

Axis of symmetry ______________

Vertex ______________

Range ______________

E. The General Quadratic Function $\{(x, y) \mid y = ax^2 + bx + c\}$

If we write the defining equation of the general quadratic function in completed square form, we are able to derive some formulas for the axis of symmetry, vertex, and range of any quadratic function.

Step 1.

$$y = ax^2 + bx + c$$

$$y = (ax^2 + bx) + c$$

Step 2.

$$y = a\left(\frac{ax^2}{a} + \frac{bx}{a}\right) + c$$

$$y = a\left(x^2 + \frac{b}{a}x\right) + c$$

Review step-by-step procedure for completing the square on page 13 of this lesson.

Step 3. The square of $\frac{1}{2}$ the numerical coefficient of x is

$$\left(\frac{1}{2} \times \frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Step 4.

$$y + \underbrace{a\left(\frac{b^2}{4a^2}\right)}_{\left[\text{Add } a\left(\frac{b^2}{4a^2}\right).\right]} = a\underbrace{\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right)}_{\left[\text{Add } a\left(\frac{b^2}{4a^2}\right).\right]} + c$$

Step 5.

$$y + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 + c$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$\boxed{y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}}$$

COMPLETED SQUARE FORM

Step 6. In this case, $k = \frac{-b}{2a}$ and $p = \frac{4ac - b^2}{4a}$.

The equation of the axis of symmetry is $x = \frac{-b}{2a}$.

The coordinates of the vertex are $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$.

GENERAL QUADRATIC FUNCTION

For the general quadratic function $\{(x, y) \mid y = ax^2 + bx + c\}$,

(1) The equation of the AXIS OF SYMMETRY of its graph is

$$x = \frac{-b}{2a}.$$

(2) The coordinates of the VERTEX of its graph are

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right).$$

(3) If $a > 0$, it has a MINIMUM value of $\frac{4ac - b^2}{4a}$ and the graph opens UPWARD.

If $a < 0$, it has a MAXIMUM value of $\frac{4ac - b^2}{4a}$ and the graph opens DOWNWARD.

(4) If $a > 0$, its RANGE is $\left\{y \mid y \geqq \frac{4ac - b^2}{4a}\right\}$.

If $a < 0$, its RANGE is $\left\{y \mid y \leqq \frac{4ac - b^2}{4a}\right\}$.

↑

MEMORIZE

These formulas can be applied to any quadratic function whose defining equation is of the form $y = ax^2 + bx + c$.

EXAMPLE: For the function $q_1 = \{(x, y) \mid y = -2x^2 + 5x - 1\}$, state the equation of the axis of symmetry of the graph, state the coordinates of the vertex of the graph, and give the range of q_1.

Solution. The defining equation is $y = -2x^2 + 5x - 1$.
For this equation, $a = -2$, $b = 5$, and $c = -1$.

$$\frac{-b}{2a} = \frac{-5}{2(-2)} = \frac{-5}{-4} = \frac{5}{4} \text{ or } 1\frac{1}{4}$$

$$\frac{4ac - b^2}{4a} = \frac{4(-2)(-1) - (5)^2}{4(-2)} = \frac{8 - 25}{-8} = \frac{-17}{-8} = \frac{17}{8} \text{ or } 2\frac{1}{8}$$

The equation of the axis of symmetry is $x = 1\frac{1}{4}$.

The coordinates of the vertex are $\left(1\frac{1}{4}, 2\frac{1}{8}\right)$.

Since $a < 0$, the range is $\{y \mid y \leqq 2\frac{1}{8}\}$.

If you know the equation of the axis of symmetry of the graph of a quadratic function, this knowledge can be used in setting up the table of values for the graph. If you choose an equal number of x values on each side of the axis of symmetry, then your parabola will have a symmetrical appearance. For example, the axis of symmetry of the graph of $\{(x, y) | y = 2x^2 - 4x\}$ is the line with equation

$$x = \frac{-b}{2a}$$

$$x = \frac{-(-4)}{2(2)}$$

$$x = 1.$$

Thus, your table of values should contain an equal number of x values on either side of x = 1.

AXIS OF SYMMETRY

x	-2	-1	0	1	2	3	4
y							

Self-correcting Exercise #5

Answers may be found on page 62 of this lesson.

1. Fill in the blanks in the following chart.

EQUATION	a	b	c	$\frac{-b}{2a}$	$\frac{4ac - b^2}{4a}$
(a) $y = 3x^2 - 7$	3	0	-7	$\frac{-0}{2(3)}$ $= 0$	$\frac{4(3)(-7) - (0)^2}{4(3)}$ $= \frac{-84}{12}$ $= -7$
(b) $y = -2x^2 + 4x$					
(c) $y = 5x^2 - 8x + 1$					

2. Using formulas, state the equation of the axis of symmetry and the coordinates of the vertex of the graph of each function. Then, determine the range of the function.

(a) $\{(x, y) \mid y = x^2 - x + 1\}$	(b) $\{(x, y) \mid y = -4x^2 + 8x + 5\}$

F. The x and y Intercepts of the Graph of a Quadratic Function

The y-intercept of the graph of a quadratic function can be determined algebraically by letting $x = 0$ in the defining equation and solving the resulting equation for y.

EXAMPLE: Find the y-intercept of the graph of $\{(x, y) \mid y = x^2 - 5\}$.
Solution. Let $x = 0$ in the defining equation $y = x^2 - 5$.

$$y = (0)^2 - 5$$
$$y = -5$$

The y-intercept is -5.

The x-intercepts of the graph of a quadratic function can be determined algebraically by letting $y = 0$ in the defining equation and solving the resulting equation for x.

EXAMPLE: Find the x-intercepts of the graph of $\{(x, y) \mid y = -2x^2 + 7x\}$.
Solution. Let $y = 0$ in the defining equation $y = -2x^2 + 7x$.

$$0 = -2x^2 + 7x$$
$$2x^2 - 7x = 0$$
$$x(2x - 7) = 0$$
$$x = 0 \quad \text{or} \quad 2x - 7 = 0$$
$$2x = 7$$
$$x = \frac{7}{2} \quad \text{or} \quad 3\frac{1}{2}$$

The x-intercepts are 0 and $3\frac{1}{2}$.

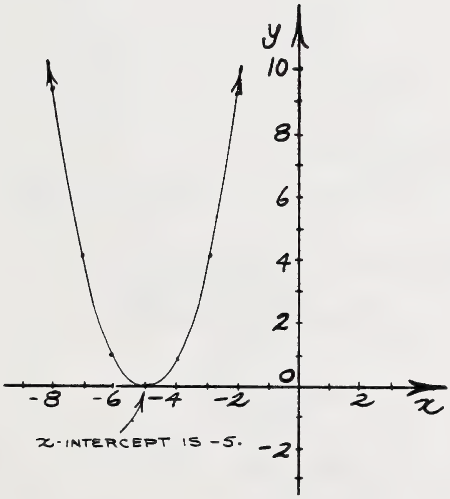
For the general quadratic function $q = \{(x, y) \mid y = ax^2 + bx + c\}$, the roots of the equation $ax^2 + bx + c = 0$ are the x-intercepts of the graph of q. You will recall that the roots of a quadratic equation may be real and equal, real and distinct, or not real. In terms of the graph of q, this means that:

(1) If the roots of $ax^2 + bx + c = 0$ are real and equal, the graph of q will come in contact with the x-axis at only one place. (i.e. the vertex of the parabola touches the x-axis.)

(2) If the roots of $ax^2 + bx + c = 0$ are real and distinct, the graph of q will cross the x-axis in two places. (i.e. each arm of the parabola will cut the x-axis.)

(3) If the roots of $ax^2 + bx + c = 0$ are not real, the graph of q will never come in contact with the x-axis. (i.e. the parabola will lie entirely above or below the x-axis.)

EXAMPLE 1: $q_1 = \{(x, y) \mid y = x^2 + 10x + 25\}$



When $y = 0$, $x^2 + 10x + 25 = 0$

$(x + 5)(x + 5) = 0$

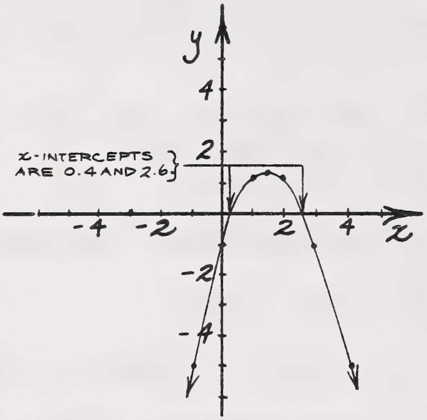
$x + 5 = 0$ or $x + 5 = 0$

$x = -5$ or $x = -5$

The x-intercept is -5.

x	-8	-7	-6	-5	-4	-3	-2
y	9	4	1	0	1	4	9

EXAMPLE 2: $q_2 = \{(x, y) \mid y = -x^2 + 3x - 1\}$



When $y = 0$, $-x^2 + 3x - 1 = 0$

$$x^2 - 3x + 1 = 0$$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{5}}{2}$$

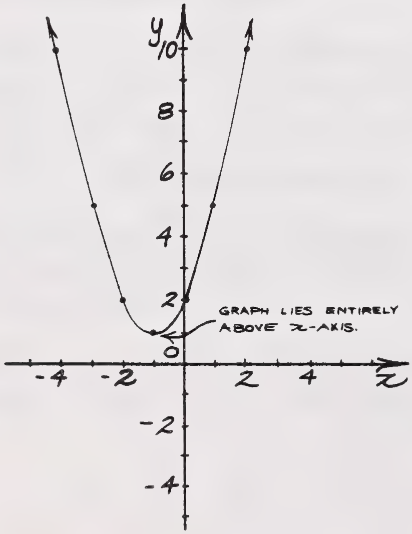
$$= \frac{3 \pm 2.236}{2}$$

$$= 2.6 \text{ or } 0.4 \text{ (approx.)}$$

The x-intercepts are 2.6 and 0.4.

x	-1	0	1	$1\frac{1}{2}$	2	3	4
y	-5	-1	1	$1\frac{1}{4}$	1	-1	-5

EXAMPLE 3: $q_3 = \{(x, y) \mid y = x^2 + 2x + 2\}$



When $y = 0$, $x^2 + 2x + 2 = 0$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 - 8}}{2}$$

$$x = \frac{-2 \pm \sqrt{-4}}{2}$$

(not real)

There are no real x-intercepts.

x	-4	-3	-2	-1	0	1	2
y	10	5	2	1	2	5	10

Self-correcting Exercise #6

Answers may be found on page 63 of this lesson.

1. Find the x and y-intercepts of the graph of each function algebraically. Then, decide if the graph comes in contact with the x-axis exactly once, if it crosses the x-axis in two places, or if it never comes in contact with the x-axis.

(a) $\{(x, y) \mid y = x^2 + x - 6\}$
Let $x = 0$.

Let $y = 0$.

(b) $\{(x, y) \mid y = 2x^2 + 12x + 18\}$

(c) $\{(x, y) \mid y = x^2 - x + 1\}$

(d) $\{(x, y) \mid y = x^2 + 5x + 1\}$

EXERCISE - QUADRATIC FUNCTIONS

1. Fill in the blanks.

(a) $\{(x, y) | x^2 + y^2 = 49\}$ is a quadratic ____________________ but is not a quadratic ____________________.

(b) If $a > 0$ in the defining equation of a quadratic function, then the vertex of its graph is the __________________ point on the graph.

(c) The graph of $\{(x, y) | y = -2x^2\}$ has ____________ as its y-intercept and _________________ as its x-intercept; that is, the graph passes through the __________________.

(d) The vertex of the graph of $\{(x, y) | y = 2(x + 3)^2 - 1\}$ occurs at the point ______________________.

(e) The range of the function $\{(x, y) | y = -(x - 1)^2 - 2\}$ is ___________________________.

(f) The axis of symmetry of the graph of a quadratic function always passes through the _____________________ of the graph.

(g) The graph of $\{(x, y) | y = 2(x + 1)^2\}$is symmetric with respect to the line whose defining equation is _________________.

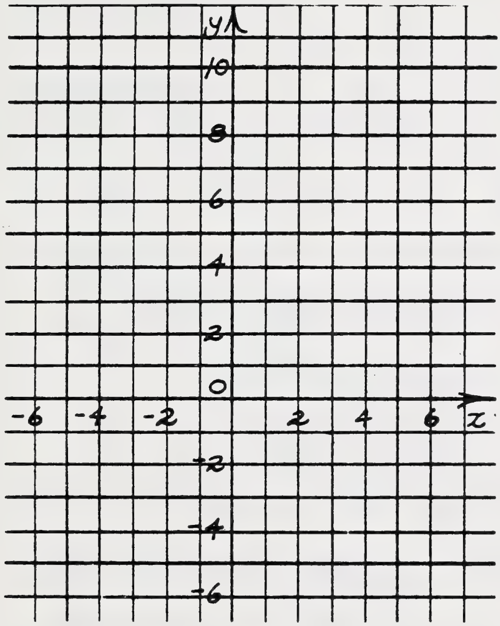
(h) The x-intercepts of the graph of $\{(x, y) | y = x^2 - 1\}$ are __________ and __________. The y-intercept is ____________.

(i) The graph of $\{(x, y) | y = x^2 + 1\}$ never comes in contact with the _______ axis.

(j) The graph of $\{(x, y) | y = -2x^2 - x\}$ opens __________________.

2. Graph the quadratic function $\{(x, y) | y = x^2 + 2x\}$. Then, answer the questions about the graph.

x	
y	



(a) The graph opens ____________.

(b) The y-intercept is ___________.

(c) The x-intercepts are ___ and ___.

(d) The equation of the axis of symmetry is _______________.

(e) The coordinates of the vertex are ________________.

(f) The range of the function is ________________.

(g) The domain of the function is ________________.

3. Decide whether each of the following statements is true or false.

	True or False?
(a) $\{(x, y) \| x^2 + y = 1\}$ is a quadratic function.	______________
(b) $\{(x, y) \| 5x^2 + y^2 = 1\}$ is a quadratic function.	______________
(c) The graph of $f(x) = x^2 + 1$ is symmetric with respect to the y-axis.	______________
(d) The graph of $y = x^2$ is a "wider" parabola than the graph of $y = \frac{1}{4}x^2$.	______________
(e) The graph of $y = -2x^2$ is a parabola which opens downward.	______________
(f) The vertex of the graph of $y = 2x^2 - 5$ is $(2, -5)$.	______________
(g) An equation of the axis of symmetry of $y = x^2 - 1$ is $x = -1$.	______________
(h) The minimum value of $\{(x, y) \| y = 3x^2 - 4\}$ is -4.	______________

True or False?

(i) The range of $y = x^2 + 2$ is $\{y | y \geqq 2\}$. ____________

(j) The graphs of $y = x^2 + 2$ and $y = -x^2 + 2$ are mirror reflections of each other in the x-axis. ____________

(k) $x^2 + 7x + \frac{49}{4} = (x + \frac{7}{2})^2$ ____________

(l) $3x^2 - x + 6 = 3(x^2 - \frac{1}{3}x) + 6$ ____________

(m) $-2x^2 + 8x + 3 = -2(x^2 + 4x) + 3$ ____________

4.

GIVEN

$q_1 = \{(x, y) | y = x^2\}$

$q_2 = \{(x, y) | y = 3x^2 - 1\}$

$q_3 = \{(x, y) | y = (x + 1)^2 - 1\}$

$q_4 = \{(x, y) | y = -3(x + 1)^2\}$

$q_5 = \{(x, y) | y = -5x^2\}$

$q_6 = \{(x, y) | y = 3(x - 1)^2 + 1\}$

$q_7 = \{(x, y) | y = x^2 + 1\}$

$q_8 = \{(x, y) | y = -(x + 1)^2\}$

For each condition below, list the functions which satisfy that condition.

EXAMPLE: x-coordinate of vertex is 0. q_1, q_2, q_5, q_7

(a) Graph opens upward. ____________

(b) y-coordinate of vertex is -1. ____________

(c) Vertex occurs at the origin. ____________

(d) Range is $\{y | y \geqq -1\}$. ____________

(e) Domain is R. ____________

(f) Equation of axis of symmetry is $x = -1$. ____________

(g) Coordinates of vertex are (0, 1). ____________

(h) Symmetric with respect to the y-axis. ____________

(i) y-coordinate of vertex is 0. ____________

(j) x-coordinate of vertex is 1. ____________

5. Put each defining equation in the completed square form $y = a(x - k)^2 + p$. Then state the equation of the axis of symmetry, the coordinates of the vertex, and the range of the function.

(a) $\{(x, y) \mid y = -2x^2 + 7\}$

$$y = -2(\qquad)^2 + \underline{\qquad}$$

Note that this is an equation

Axis of symmetry is $x = \underline{\qquad}$.

Coordinates of vertex are $(\qquad)$.

Range is $\{y \mid \qquad\}$.

(b) $\{(x, y) \mid y = 7x^2\}$

Axis of symmetry is ________.

Coordinates of vertex are ________.

Range is ____________________.

(c) $\{(x, y) \mid y = 3x^2 + 6x\}$

$$y = 3(x^2 + 2x)$$
$$y + 3(\underline{\quad}) = 3(x^2 + 2x + \underline{\quad})$$
$$y + \underline{\quad} = 3(\qquad)^2$$
$$y = 3(\qquad)^2 - \underline{\quad}$$

Axis of symmetry is ________.

Coordinates of vertex are _____.

Range is ____________________.

(d) $\{(x, y) \mid y = x^2 + x - 7\}$

$$y = (x^2 + x) - 7$$
$$y + \underline{\quad} = (x^2 + x + \underline{\quad}) - 7$$
$$y + \underline{\quad} = (\qquad)^2 - 7$$
$$y = (\qquad)^2 - \underline{\quad}$$

Axis of symmetry is ________.

Coordinates of vertex are _____.

Range is ____________________.

(e) $\{(x, y) \mid y = 2x^2 - 8x + 3\}$

Axis of symmetry is ________.

Coordinates of vertex are _____.

Range is ____________________.

(f) $\{(x, y) \mid y = -2x^2 + 3x + 1\}$

Axis of symmetry is ________.

Coordinates of vertex are _____.

Range is ____________________.

6. Using formulas, find the equation of the axis of symmetry, the coordinates of the vertex and the range of each function.

(a) $\{(x, y) \mid y = 2x^2 - 4\}$

a = ___, b = ___, c = ___

$\frac{-b}{2a} =$

$\frac{4ac - b^2}{4a} =$

Axis of symmetry is ___________.

Coordinates of vertex are _____.

Range is ____________________.

(b) $\{(x, y) \mid y = -x^2 - x\}$

a = ___, b = ___, c = ___

Axis of symmetry is ___________.

Coordinates of vertex are _____.

Range is ____________________.

(c) $\{(x, y) \mid y = -5x^2\}$

a = ___, b = ___, c = ___

Axis of symmetry is ___________.

Coordinates of vertex are _____.

Range is ____________________.

(d) $\{(x, y) \mid y = x^2 - 6x + 8\}$

a = ___, b = ___, c = ___

Axis of symmetry is ___________.

Coordinates of vertex are _____.

Range is ____________________.

(e) $\{(x, y) \mid y = 2x^2 - 8x + 3\}$

a = ___, b = ___, c = ___

Axis of symmetry is ___________.

Coordinates of vertex are _____.

Range is ____________________.

(f) $\{(x, y) \mid y = -2x^2 + 3x + 1\}$

a = ___, b = ___, c = ___

Axis of symmetry is ___________.

Coordinates of vertex are _____.

Range is ____________________.

7. Find the x and y-intercepts of the graph of each function. Then, discuss the number of times each graph will come in contact with the x-axis.

(a) $\{(x, y) \mid y = x^2 - 4\}$ (b) $\{(x, y) \mid y = x^2 - 2x + 1\}$

(c) $\{(x, y) \mid y = x^2 + 3x - 1\}$ (d) $\{(x, y) \mid y = x^2 - x + 5\}$

8. A bullet is fired directly upward with an initial velocity of 120 ft. per second. Its distance s feet above the ground after a time of t seconds has elapsed is given by $s = 120t - 16t^2$. What is the maximum height reached by the bullet and how long does it take to reach this height? (Hint: Use the formulas for the coordinates of the vertex of a parabola.)

For the equation $s = 120t - 16t^2$, a = ___, b = ___ and c = ___.

At a maximum, $t = \dfrac{-b}{2a} =$

$$s = \frac{4ac - b^2}{4a} =$$

The maximum height reached by the bullet is __________ feet and it takes

__________ seconds to reach this height.

Topic Two: Other Quadratic Relations

So far, we have dealt only with quadratic relations that have defining equations of the form $y = ax^2 + bx + c$ where $a, b, c \in R$. The graphs of all such relations are parabolas that open either upward or downward in the coordinate plane.

-If "a" is positive, the parabola opens ________________.

-If "a" is negative, the parabola opens ________________.

A parabola defined by $y = ax^2 + bx + c$ is symmetric about the line $x = \frac{-b}{2a}$ and has a vertex that occurs at $\left(\frac{-b}{2a}, \frac{\qquad}{4a}\right)$. Its domain is R and its range is $\{y \mid y \geqq \frac{4ac - b^2}{4a}\}$ if $a > 0$ and $\{y \leqq \frac{\qquad}{4a}\}$ if $a < 0$.

Any quadratic relation of the form $\{(x, y) \mid y = ax^2 + bx + c\}$ is a function, since for each value of x, there is one and only one value of y. Also if we apply the vertical line test to a parabola that opens up or down, we find that any vertical line will cut the parabola at only one point.

In the sections that follow, we will discuss some other types of quadratic relations.

A. Relations of the Form $\{(x, y) \mid x = ay^2 + by + c\}$

EXAMPLE: $\boxed{\{(x, y) \mid x = y^2 - 2y - 8\}}$

x-intercepts

Let $y = 0$.
$x = (0)^2 - 2(0) - 8$
$x = 0 - 0 - 8$
$x = -8$
The graph cuts the x-axis at -8.

y-intercepts

Let $x = 0$.
$0 = y^2 - 2y - 8$
$0 = (y - 4)(y + 2)$
$y = 4$ or $y = -2$
The graph cuts the y-axis at 4 and -2.

Domain

Make y the subject.
$y^2 - 2y = x + 8$
Complete the square of $y^2 - 2y$ by adding 1 to both sides.
$y^2 - 2y + 1 = x + 8 + 1$
$(y - 1)^2 = x + 9$
$y - 1 = \pm\sqrt{x + 9}$
$y = 1 \pm \sqrt{x + 9}$
y is real when
$x + 9 \geqq 0$
$x \geqq -9$
Domain is $\{x \mid x \geqq -9\}$.

Since the domain is $\{x \mid x \geqq -9\}$, we can use any values of x that are greater than or equal to -9 in our table of values.

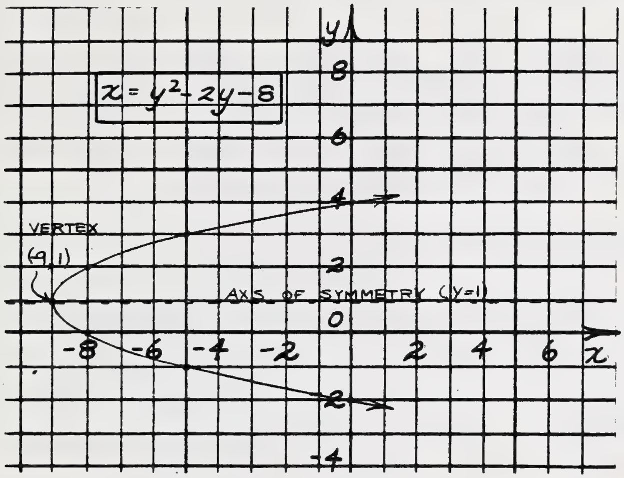
$\boxed{y = 1 \pm \sqrt{x + 9}}$

When $x = -9$, $y = 1 \pm \sqrt{-9 + 9} = 1 \pm \sqrt{0} = 1 \pm 0 = 1$

When $x = -8$, $y = 1 \pm \sqrt{-8 + 9} = 1 \pm \sqrt{1} = 1 \pm 1 = 2$ or 0

When $x = -5$, $y = 1 \pm \sqrt{-5 + 9} = 1 \pm \sqrt{4} = 1 \pm 2 = 3$ or ____

When $x = 0$, $y = 1 \pm \sqrt{0 + 9} = 1 \pm \sqrt{9} = 1 \pm 3 =$ ____ or ____



Fill in the blanks in the table of values below.

x	-9	-8		-5		0	
y	1					4	-2

If we plot the ordered pairs from the table, we find that the graph is a parabola.

Note that:

1. The defining equation of this relation is $x = y^2 - 2y - 8$. If we compare this equation with the general form $x = ay^2 + by + c$, we see that in this case $a = 1$, $b = -2$, and $c = -8$.

2. The graph opens to the right (and "a" is positive).

3. The vertex of the parabola occurs at (-9, 1). (See graph.) The value of x at the vertex can be determined by the formula:

$$x = \frac{4ac - b^2}{4a} = \frac{4(1)(-8) - (-2)^2}{4(1)} = \frac{-\quad}{4} =$$

The value of y at the vertex can be determined by the formula:

$$y = \frac{-b}{2a} = \frac{-(-2)}{2(1)} =$$

(Note that these formulas for x and y are the reverse of what they are for relations of the form $y = ax^2 + bx + c$.)

4. The equation of the axis of symmetry is $y = 1$. (See graph.) It can be determined by the formula $y = \frac{-b}{2a}$. (Note that the axis of symmetry is a horizontal line while for relations of the form $y = ax^2 + bx + c$ it is always a vertical line.)

5. The domain of the relation is $\{x \mid x \geqq -9\}$. The range is R.

6. The relation is not a function.
   (i) y^2 occurs in the defining equation.
   (ii) Some vertical lines cut the graph in more than one place.

In general, for any relation of the form $\{(x, y) \mid x = ay^2 + by + c\}$ where $a, b, c \in R$:

1. The graph is a parabola that opens to the right if "a" is positive and to the left if "a" is negative.
2. The vertex occurs at $\left(\frac{4ac - b^2}{4a}, \frac{-b}{2a}\right)$.
3. The equation of the axis of symmetry is $y = \frac{-b}{2a}$.
4. The domain is $\{x \mid x \geqq \frac{4ac - b^2}{4a}\}$ if $a > 0$ and $\{x \mid x \leqq \frac{4ac - b^2}{4a}\}$ if $a < 0$. The range is R. The relation is not a function.

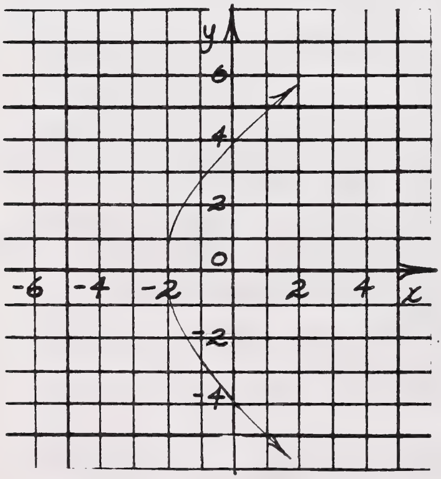
* Compare the characteristics of $\{(x, y) \mid x = ay^2 + by + c\}$ that are listed above with the characteristics of the general quadratic function $\{(x, y) \mid y = ax^2 + bx + c\}$ listed on page 20 of this lesson.

Self-correcting Exercise #7

Answers may be found on page 64 of this lesson.

1. For the graph of each relation, tell whether it opens to the left or right. Then give the equation of the axis of symmetry, the coordinates of the vertex, the domain, range and x and y-intercepts.

(a)



(i) Right or Left? ________________

(ii) Axis of Symmetry ________________

(iii) Coordinates of Vertex ________________

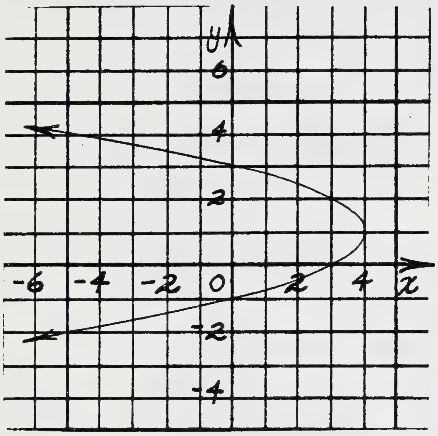
(iv) Domain ________________

(v) Range ________________

(vi) x-intercepts ________________

(vii) y-intercepts ________________

(b)



(i) Right or Left? ________

(ii) Axis of Symmetry ________

(iii) Coordinates of Vertex ________

(iv) Domain ________

(v) Range ________

(vi) x-intercepts ________

(vii) y-intercepts ________

2. Use formulas to find the equation of the axis of symmetry, the coordinates of the vertex and the domain of each relation.

(a) $\{(x,y) | x = 3y^2 - 5\}$

$a =$ ___, $b =$ ___, $c =$ ___

$$\frac{4ac - b^2}{4a}$$

=

$$\frac{-b}{2a}$$

=

Axis of symmetry is ________.

Coordinates of vertex are ______.

Domain is ________________.

(b) $\{(x,y) | x = -y^2 + 5y - 3\}$

$a =$ ___, $b =$ ___, $c =$ ___

Axis of symmetry is ________.

Coordinates of vertex are ______.

Domain is ________________.

B. Relations of the Form $\{(x, y) \mid x^2 + y^2 = r^2\}$

EXAMPLE: $\boxed{\{(x, y) \mid x^2 + y^2 = 25\}}$

x-intercepts

Let $y = 0$.
$x^2 + (0)^2 = 25$
$x^2 = 25$
$x = \pm 5$
x-intercepts are 5 and -5.

y-intercepts

Let $x = 0$.
$(0)^2 + y^2 = 25$
$y^2 = 25$
$y = \pm 5$
y-intercepts are 5 and -5.

Domain

Make y the subject.
$y^2 = 25 - x^2$
$y = \pm\sqrt{25 - x^2}$
y is real when
$25 - x^2 \geqq 0$
$-x^2 \geqq -25$
$x^2 \leqq 25$
$|x| \leqq 5$
$-5 \leqq x \leqq 5$
Domain is $\{x \mid -5 \leqq x \leqq 5\}$.

Since the domain is $\{x \mid -5 \leqq x \leqq 5\}$, we can use any values of x between -5 and 5 inclusive in the table of values.

$\boxed{y = \pm\sqrt{25 - x^2}}$

When $x = 0$, $y = \pm\sqrt{25 - 0} = \pm\sqrt{25} = \pm$ ___

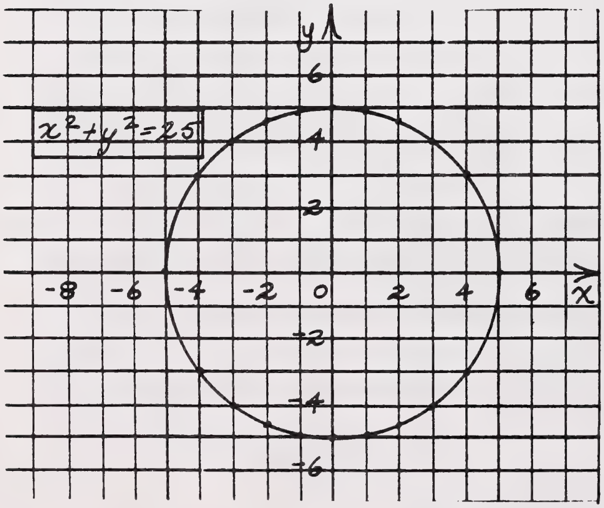
When $x = \pm 1$, $y = \pm\sqrt{25 - (\pm 1)^2} = \pm\sqrt{25 - 1} = \pm\sqrt{24} \doteq \pm$ 4.9 (SEE TABLE OF SQUARE ROOTS IN LESSONB)

When $x = \pm 2$, $y = \pm\sqrt{25 - (\pm 2)^2} = \pm\sqrt{25 - 4} = \pm\sqrt{21} \doteq \pm$ ___

When $x = \pm 3$, $y = \pm\sqrt{25 - (\pm 3)^2} = \pm\sqrt{25 - \underline{\quad}} = \pm\sqrt{\underline{\quad}} = \pm$ ___

When $x = \pm 4$, $y = \pm\sqrt{25 - (\pm 4)^2} = \pm\sqrt{\qquad\qquad} = \pm\sqrt{\underline{\quad}} = \pm$ ___

When $x = \pm 5$, $y = \pm\sqrt{25 - (\pm 5)^2} = \pm\sqrt{25 - \underline{\quad}} = 0$



Fill in the blanks in the table of values below.

x	0	±1	±2	±3	±4	±5
y	±5			±4		0

If we plot the ordered pairs from the table, we find that the graph is a circle centred at the origin.

With reference to the circle graph at the bottom of page 36, note that:

1. The defining equation of the relation is $x^2 + y^2 = 25$. If we compare this equation with the general form $x^2 + y^2 = r^2$, we see that in this case $r = 5$. The value of r corresponds to the radius of the circle.

2. A compass was used to draw this circle graph. (The legs of the compass were placed 5 units apart and the compass was centred at the origin.)

3. This relation is not a function since y^2 occurs in the defining equation and its graph does not pass the vertical line test.

> In general, for any relation of the form $\{(x, y) \mid x^2 + y^2 = r^2\}$ where r is a positive real number, the graph is a circle with a radius of r units. This circle can be drawn using a compass, centred at the origin, whose legs are r units apart. The relation is not a function.

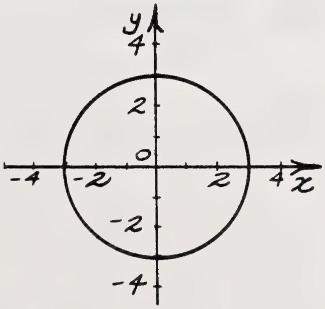
Self-correcting Exercise #8

Answers may be found on page 64 of this lesson.

1. Put each equation in the form $x^2 + y^2 = r^2$. Then state the radius of the circle it represents.

	Form $x^2 + y^2 = r^2$	Radius of Circle
(a) $x^2 - 8 + y^2 = 0$	$x^2 + y^2 = 8$	$\sqrt{8} \doteq 2.8$
(b) $y^2 = 49 - x^2$		
(c) $4x^2 = 1 - 4y^2$		
(d) $\frac{x^2 + y^2}{5} = 1$		
(e) $5x^2 + 5y^2 = 45$		

2. For the circle graph below, give its radius, domain, range, x-intercepts and y-intercepts.



(a) Radius ______________

(b) Domain ______________

(c) Range ______________

(d) x-intercepts ______________

(e) y-intercepts ______________

IF $a = b$, THE GRAPH WOULD BE A CIRCLE.

C. Relations of the Form $\{(x, y) \mid ax^2 + by^2 = c\}$ where $a, b, c \in {}^{+}R$ and $a \neq b$

EXAMPLE: $\boxed{\{(x, y) \mid 4x^2 + 9y^2 = 36\}}$

NOTE THAT THESE COEFFICIENTS ARE POSITIVE.

x-intercepts

Let $y = 0$.

$4x^2 + 9(0)^2 = 36$

$4x^2 + 0 = 36$

$4x^2 = 36$

$x^2 = 9$

$x = \pm 3$

x-intercepts are 3 and -3.

y-intercepts

Let $x = 0$.

$4(0)^2 + 9y^2 = 36$

$0 + 9y^2 = 36$

$9y^2 = 36$

$y^2 = 4$

$y = \pm 2$

y-intercepts are 2 and -2.

Domain

Make y the subject.

$9y^2 = 36 - 4x^2$

$y^2 = \frac{36 - 4x^2}{9}$

$y = \frac{\pm\sqrt{36 - 4x^2}}{3}$

y is real when

$36 - 4x^2 \geqq 0$

$-4x^2 \geqq -36$

$x^2 \leqq 9$

$|x| \leqq 3$

$-3 \leqq x \leqq 3$

Domain is $\{x \mid -3 \leqq x \leqq 3\}$.

Since the domain is $\{x \mid -3 \leqq x \leqq 3\}$, we can use any values of x between -3 and 3 inclusive in the table of values.

$$y = \boxed{\frac{\pm\sqrt{36 - 4x^2}}{3}}$$

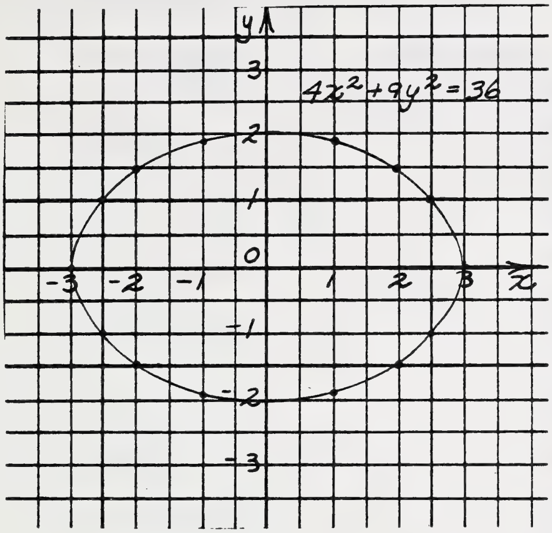
When $x = 0$, $y = \frac{\pm\sqrt{36 - 4(0)^2}}{3} = \frac{\pm\sqrt{36 - 0}}{3} = \frac{\pm 6}{3} = \pm\underline{\qquad}$

When $x = \pm 1$, $y = \frac{\pm\sqrt{36 - 4(\pm 1)^2}}{3} = \frac{\pm\sqrt{36 - 4(1)}}{3} = \frac{\pm\sqrt{36 - 4}}{3} = \frac{\pm\sqrt{32}}{3} \doteq \frac{\pm 5.657}{3} = \pm 1.9$

When $x = \pm 2$, $y = \frac{\pm\sqrt{36 - 4(\pm 2)^2}}{3} = \frac{\pm\sqrt{36 - 4(4)}}{3} = \frac{\pm\sqrt{36 - 16}}{3} = \frac{\pm\sqrt{\qquad}}{3} \doteq \pm\underline{\qquad}$

When $x = \pm 2.5$, $y = \frac{\pm\sqrt{36 - 4(\pm 2.5)^2}}{3} = \frac{\pm\sqrt{36 - 4(6.25)}}{3} = \frac{\pm\sqrt{36 - 25}}{3} = \frac{\pm\sqrt{\quad}}{3} \doteq \frac{\pm\underline{\quad}}{3} = \pm\underline{\qquad}$

When $x = \pm 3$, $y = \frac{\pm\sqrt{36 - 4(\pm 3)^2}}{3} = \frac{\pm\sqrt{36 - 4(9)}}{3} =$



Fill in the blanks in the table of values below.

x	0	±1	±2	±2.5	±3
y	±2		±1.5		

If we plot the ordered pairs from the table, we find that the graph is an ellipse centred at the origin. Note that the ellipse lies between -3 and 3 inclusive on the x-axis and -2 and 2 inclusive on the y-axis. This relation is not a function since y^2 occurs in the defining equation and its graph does not pass the vertical line test.

In general, for any relation of the form $\{(x, y) \mid ax^2 + by^2 = c\}$ where a, b, and c are positive real numbers and $a \neq b$, the graph is an ellipse centred at the origin. The relation is not a function.

Self-correcting Exercise #9

Answers may be found on page 65 of this lesson.

1. Put each equation in the form $ax^2 + by^2 = c$ where $a, b, c \in {}^{+}R$.

(a) $-2y^2 = -5 + 3x^2$

____

(b) $x^2 + 2y^2 - 1 = 0$

____

(c) $4y^2 = 3 - 7x^2$

____

(d) $5x^2 = 8 - 3y^2$

____

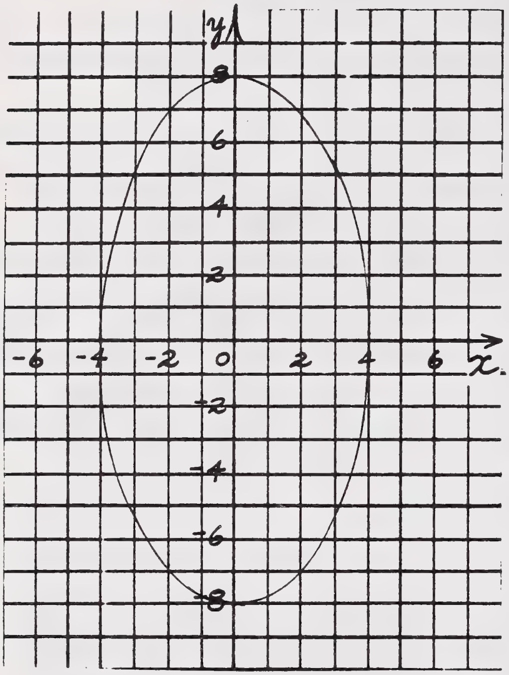
(e) $5x^2 - 7 = -4y^2$

____

(f) $9 - 4x^2 = 3y^2$

____

2. For the ellipse below, give its domain, range, x-intercepts and y-intercepts.



(a) Domain

____

(b) Range

____

(c) x-intercepts

____

(d) y-intercepts

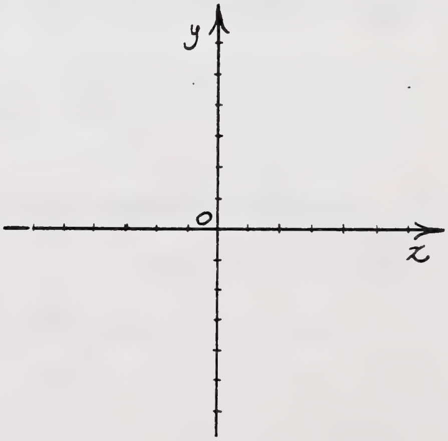
____

3. For each equation, determine the x and y-intercepts of the graph. Then sketch the graph, showing the intercepts and general shape. (Tables of values are not required.)

(a) $25x^2 + 16y^2 = 400$

x-intercepts	y-intercepts
Let $y = 0$.	Let $x = 0$.

Sketch



(b) $4x^2 + 9y^2 = 36$

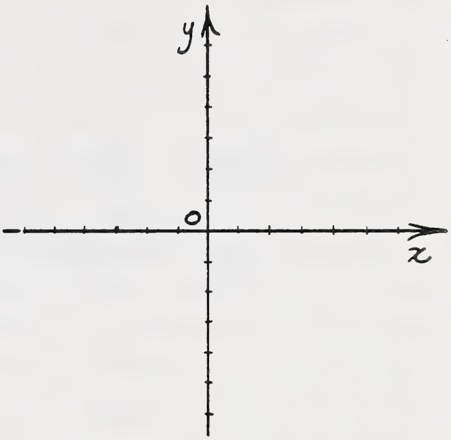
x-intercepts

Let $y = 0$.

y-intercepts

Let $x = 0$.

Sketch



D. Relations of the Form $\{(x, y) \mid ax^2 - by^2 = \pm c\}$ where $a, b, c \in {}^{+}R$.

EXAMPLE 1: $\boxed{\{(x, y) \mid x^2 - y^2 = 16\}}$

THE TWO TERMS ARE SUBTRACTED.

RIGHT SIDE IS POSITIVE.

x-intercepts

Let $y = 0$.

$x^2 - (0)^2 = 16$

$x^2 = 16$

$x = \pm$____

x-intercepts are

______ and ______.

y-intercepts

Let $x = 0$.

$(0)^2 - y^2 = 16$

$-y^2 = 16$

$y^2 = -16$

$y = \pm\sqrt{-16}$

(not real)

There are no y-intercepts.

Domain

Make y the subject.

$-y^2 = 16 - x^2$

$y^2 = x^2 - 16$

$y = \pm\sqrt{x^2 - 16}$

y is real when

$x^2 - 16 \geqq 0$

$x^2 \geqq 16$

$|x| \geqq 4$

(i.e. $x \geqq 4$ or $x \leqq -4$)

Domain is $\{x \mid x \geqq 4 \text{ or } x \leqq -4\}$

Since the domain is $\{x \mid x \geqq 4 \text{ or } x \leqq -4\}$, we can use any values of x that are numerically greater than or equal to 4 in the table of values.

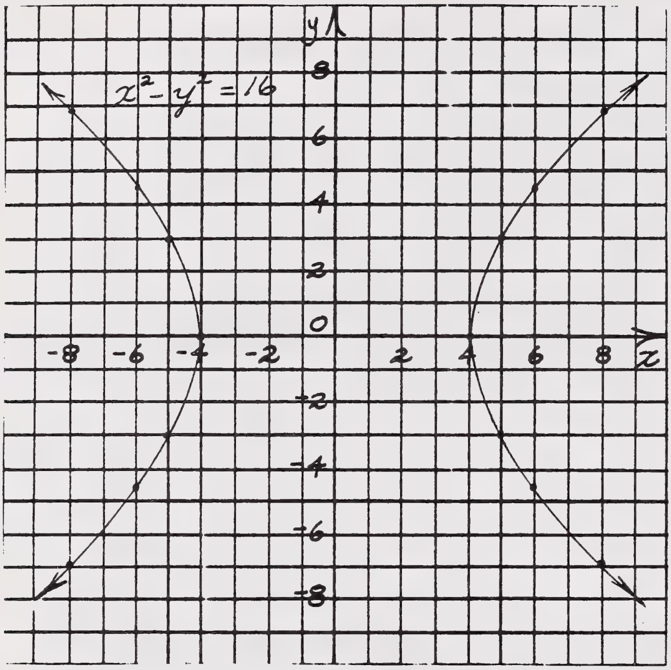
$$\boxed{y = \pm\sqrt{x^2 - 16}}$$

When $x = \pm 4$, $y = \pm\sqrt{(\pm 4)^2 - 16} = \pm\sqrt{16 - 16} =$ ______

When $x = \pm 5$, $y = \pm\sqrt{(\pm 5)^2 - 16} = \pm\sqrt{25 - 16} = \pm\sqrt{\_\_\_\_} = \pm$ ______

When $x = \pm 6$, $y = \pm\sqrt{(\pm 6)^2 - 16} = \pm\sqrt{36 - 16} = \pm\sqrt{20} \doteq \pm 4.5$

When $x = \pm 8$, $y = \pm\sqrt{(\pm 8)^2 - 16} = \pm\sqrt{64 - 16} = \pm\sqrt{\_\_\_\_} \doteq \pm$ ______



Fill in the blanks in the table of values below.

x	±4	±5	±6	±8
y	0		±4.5	

If we plot the ordered pairs from the table, we find that the graph is a hyperbola centred at the origin. Note that the hyperbola consists of two branches that lie on either side of y-axis. The hyperbola cuts the x-axis in two places but never comes in contact with y-axis.

This relation is not a function since y^2 occurs in the defining equation and its graph doesn't pass the vertical line test.

EXAMPLE 2: $\boxed{\{(x, y) \mid x^2 - 4y^2 = -16\}}$

THE TWO TERMS ARE SUBTRACTED.

RIGHT SIDE IS NEGATIVE.

x-intercepts

Let y = 0.

$$x^2 - 4(0)^2 = -16$$
$$x^2 - 0 = -16$$
$$x^2 = -16$$
$$x = \pm\sqrt{-16}$$
(not real)

There are no x-intercepts.

y-intercepts

Let x = 0.

$$(0)^2 - 4y^2 = -16$$
$$-4y^2 = -16$$
$$y^2 = \underline{\quad\quad}$$
$$y = \underline{\quad\quad}$$

The y-intercepts

are _____ and _____.

Domain

Make y the subject.

$$-4y^2 = -x^2 - 16$$
$$4y^2 = x^2 + 16$$
$$y^2 = \frac{x^2 + 16}{4}$$
$$y = \frac{\pm\sqrt{x^2 + 16}}{2}$$

y is real when
$x^2 + 16 \geqq 0$.

This is an unconditional inequality that is true for all real values of x. The domain is R.

Since the domain is R, we can use any real number for x in the table of values.

$$\boxed{y = \frac{\pm\sqrt{x^2 + 16}}{2}}$$

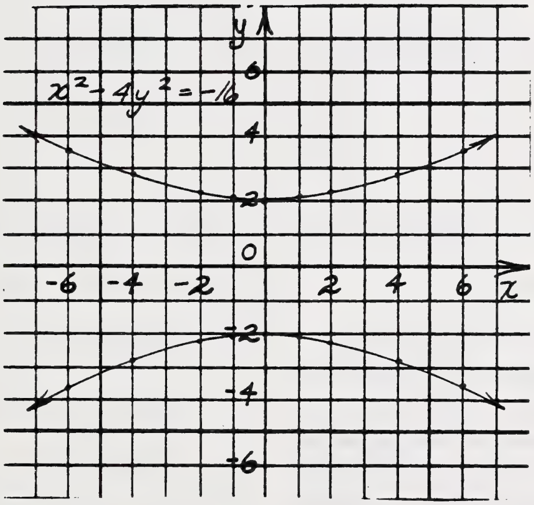
When $x = 0$, $y = \frac{\pm\sqrt{(0)^2 + 16}}{2} = \frac{\pm\sqrt{16}}{2} = \frac{\pm\underline{\quad}}{2} = \underline{\qquad}$

When $x = \pm1$, $y = \frac{\pm\sqrt{(\pm1)^2 + 16}}{2} = \frac{\pm\sqrt{1 + 16}}{2} = \frac{\pm\sqrt{17}}{2} \doteq \frac{\pm4.123}{2} = \pm2.1$

When $x = \pm2$, $y = \frac{\pm\sqrt{(\pm2)^2 + 16}}{2} = \frac{\pm\sqrt{4 + 16}}{2} = \frac{\pm\sqrt{\underline{\quad}}}{2} \doteq \frac{\pm\underline{\qquad}}{2} = \pm\underline{\qquad}$

When $x = \pm4$, $y = \frac{\pm\sqrt{(\pm4)^2 + 16}}{2} =$

When $x = \pm6$, $y = \frac{\pm\sqrt{(\pm6)^2 + 16}}{2} =$



Fill in the blanks in the table of values below.

x	0	±1	±2	±4	±6
y		±2.1			

Again, the graph is a hyperbola centred at the origin. But, in this case, the two branches of the hyperbola lie on either side of the x-axis. The hyperbola cuts the y-axis in two places but never comes in contact with the x-axis.

This relation is not a function since y^2 occurs in the defining equation and its graph doesn't pass the vertical line test.

In general, for any relation of the form $\{(x, y) | ax^2 - by^2 = \pm c\}$ where a, b and c are positive real numbers, the graph is a hyperbola centred at the origin. If the defining equation is of the form $ax^2 - by^2 = c$, the two branches of the hyperbola cut the x-axis. If it is of the form $ax^2 - by^2 = -c$, the branches cut the y-axis. The relation is not a function.

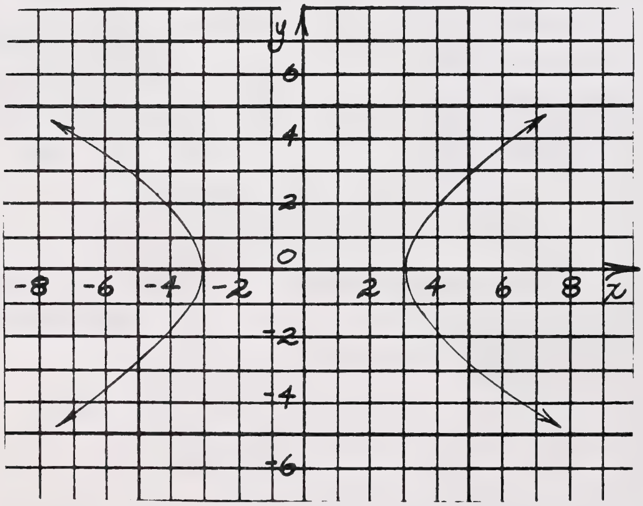
Self-correcting Exercise #10

Answers may be found on page 66 of this lesson.

1. Put each equation in the form $ax^2 - by^2 = \pm c$ where $a, b, c \in {}^{+}R$. Then state whether the branches of the hyperbola defined by the given equation will cut the x or y-axis.

	Form $ax^2 - by^2 = \pm c$	Axis Cut
(a) $5y^2 = 3x^2 - 2$	$3x^2 - 5y^2 = 2$	x-axis
(b) $2x^2 - 3y^2 + 1 = 0$		
(c) $x^2 = 3y^2 - 7$		
(d) $3y^2 - 2x^2 = -7$		
(e) $y^2 + 5 = 2x^2$		

2. For the hyperbola given below, give its domain, range, x-intercepts and y-intercepts.



(a) Domain ____________

(b) Range ____________

(c) x-intercepts ____________

(d) y-intercepts ____________

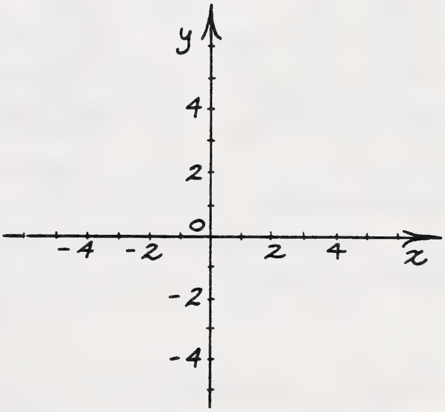
3. For each graph, determine the x and y-intercepts of the graph and evaluate y when x = ±4. Then sketch the graph, showing the x and y-intercepts and the general shape.

(a) $2x^2 - 5y^2 = 8$

Let $y = 0$.

Let $x = 0$.

When $x = \pm 4$,

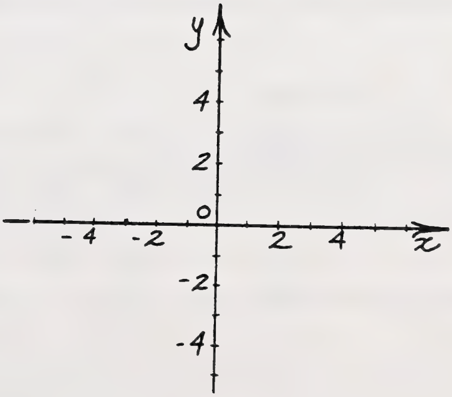


(b) $x^2 - 9y^2 = -81$

Let $y = 0$.

Let $x = 0$.

When $x = \pm 4$,



E. Relations of the Form $\{(x, y) | xy = k\}$ where $k \in R$

Note that any equation of the form $xy = k$ is a quadratic equation since the term xy is of degree 2. Thus, $\{(x, y) | xy = k\}$ is a quadratic relation.

EXAMPLE 1: $\boxed{\{(x, y) | xy = 24\}}$

x-intercepts	y-intercepts	Domain
Let $y = 0$. $x(0) = 24$ There is no real value of x which satisfies this equation. Thus, there are no x-intercepts.	Let $x = 0$. $(0)y = 24$ There is no real value of y which satisfies this equation. Thus, there are no y-intercepts.	Make y the subject. $y = \frac{24}{x}$ y is real for any real value of x except zero (since $\frac{24}{0}$ is undefined).

In setting up a table of values, we can assign any real value to x (except zero).

$$\boxed{y = \frac{24}{x}}$$

When $x = -8$, $y = \frac{24}{-8} =$ ______

When $x = -6$, $y =$

When $x = -4$, $y =$

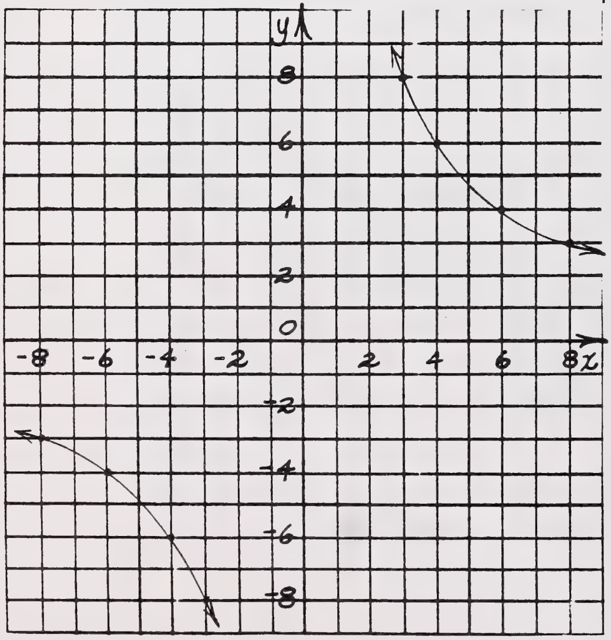
When $x = -3$, $y =$

When $x = 8$, $y = \frac{24}{8} =$ ______

When $x = 6$, $y =$

When $x = 4$, $y =$

When $x = 3$, $y =$



Fill in the blanks in the table below.

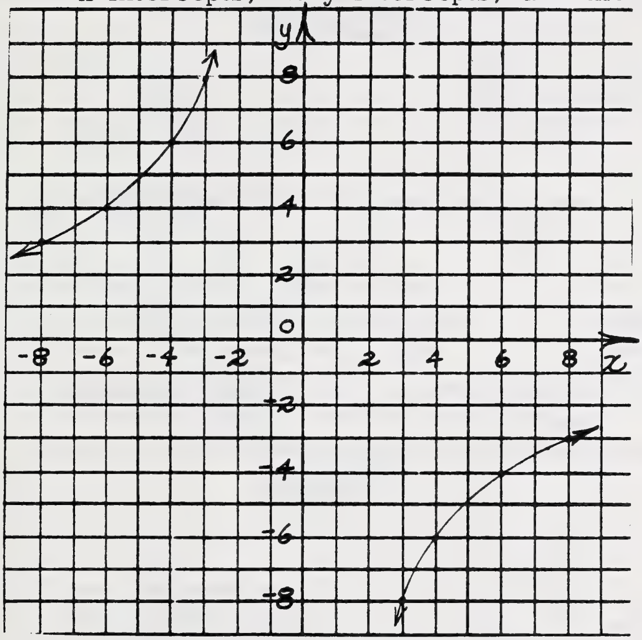
x	-8	-6	-4	-3	3	4	6	8
y	-3							3

If we plot the ordered pairs from the table, we find that the graph is a hyperbola. Note that the two branches of the hyperbola lie in Quadrants I and III and never come in contact with either the x or y-axis.

The relation is a function since for each non-zero value of x there is one and only one value of y. (Note that its graph passes the vertical line test.)

EXAMPLE 2: $\boxed{\{(x, y) \mid xy = -24\}}$

As with the relation $\{(x,y \mid xy = 24\}$, the graph of this relation has no x-intercepts, no y-intercepts, and the domain of the relation consists of all non-zero real values of x. Fill in the blanks in the table below.



x	-8	-6	-4	-3	3	4	6	8
y	3							-3

Again, the graph is a hyperbola that never comes in contact with the x or y-axis. But, in this case, the branches of the hyperbola lie in Quadrants II and IV.

This relation is also a function since for each non-zero value of x, there is one and only one value of y. (Note that its graph passes the vertical line test.)

In general, for any relation of the form $\{(x, y) \mid xy = k\}$ where k is a real number, the graph is a hyperbola. If k is positive, the two branches of the hyperbola will lie in Quadrants I and III. If k is negative, the two branches will lie in Quadrants II and IV.

Self-correcting Exercise #11

Answers may be found on page 67 of this lesson.

1. Tell whether the branches of each hyperbola lie in Quadrants I and III or Quadrants II and IV.

Quadrants

(a) $xy = -4$ Quadrants II and IV

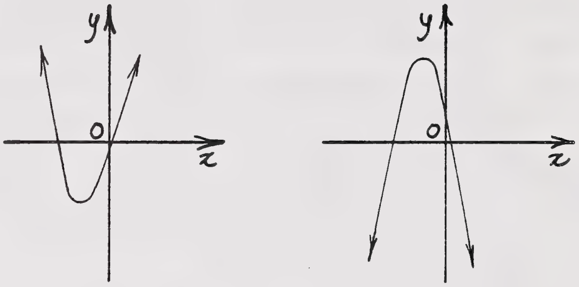
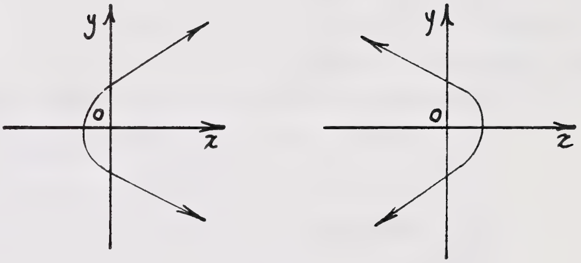
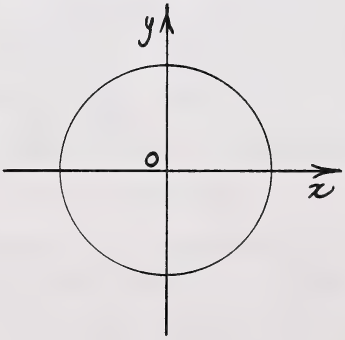
(b) $xy = 7$ ____________

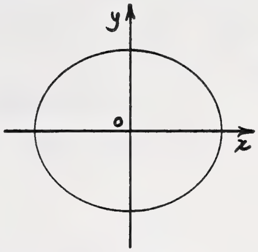
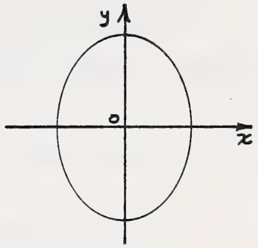
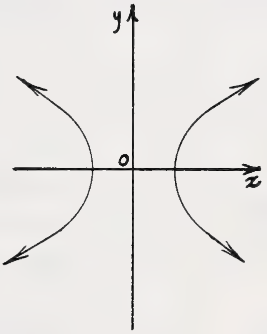
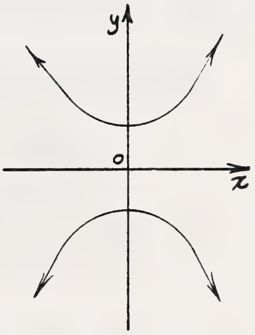
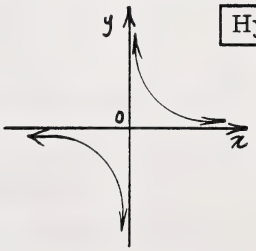
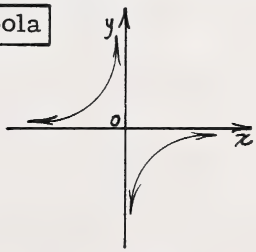
(c) $x = \frac{3}{y}$ ____________

(d) $-y = \frac{5}{x}$ ____________

(e) $-x = \frac{-6}{y}$ ____________

F. Summary - Some Quadratic Relations

DEFINING EQUATION	GRAPH
$\boxed{y = ax^2 + bx + c}$ where $a, b, c \in R$ Note that y is to the first degree and x is to the second degree.	Parabola  Opens upward when $a > 0$. Opens downward when $a < 0$.
$\boxed{x = ay^2 + by + c}$ where $a, b, c \in R$ Note that x is to the first degree and y is to the second degree.	Parabola  Opens to the right when $a > 0$. Opens to the left when $a < 0$.
$\boxed{x^2 + y^2 = r^2}$ where $r \in {}^{+}R$ Both x and y are to the second degree. x^2 and y^2 both have coefficients of 1.	Circle  Circle centred at the origin.

DEFINING EQUATION	GRAPH	
$\boxed{ax^2 + by^2 = c}$ where $a, b, c \in {}^{+}R$ and $a \neq b$. Both x and y are to the second degree. Note that the coefficients of x^2 and y^2 are positive but numerically different.	Ellipse  x-intercepts are numerically larger than y-intercepts.	 y-intercepts are numerically larger than x-intercepts.
$\boxed{ax^2 - by^2 = \pm c}$ where $a, b, c \in {}^{+}R$ Both x and y are to the second degree. Note that the coefficients of x^2 and y^2 have opposite signs.	Hyperbola  c is positive. Branches of hyperbola intersect x-axis.	 c is negative. Branches of hyperbola intersect y-axis.
$\boxed{xy = k}$ where $k \in R$ Both x and y are to the first degree and are multiplied to obtain the constant k.	Hyperbola  k is positive. Branches of hyperbola lie in Quadrants I and III.	 k is negative. Branches of hyperbola lie in Quadrants II and IV.

EXERCISE - Some Quadratic Relations

1. Put each equation in standard form. (See defining equations on pages 48 and 49 of this lesson.) Then state whether the graph of the equation is a parabola, circle, ellipse, or hyperbola.

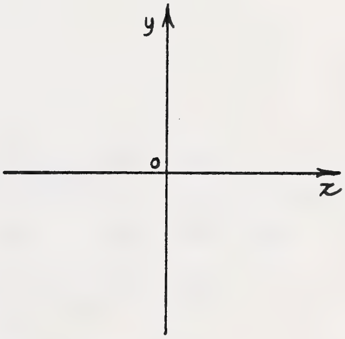
	Standard Form	Graph
(a) $3x^2 = 9 - 3y^2$	$x^2 + y^2 = 3$	circle
(b) $3x^2 = 3y^2 + 1$		
(c) $3x^2 = 1 - 3y^2$		
(d) $5xy = 125$		
(e) $-5x^2 - 7y^2 = -25$		
(f) $x + 6 = y^2 - y$	$x = y^2 - y - 6$	parabola
(g) $y - 2x^2 = 0$		
(h) $x = \frac{-16}{y}$		
(i) $4x^2 = 16 - 4y^2$		
(j) $y - 4x = 2x^2$		
(k) $4x^2 = y^2 + 12$		
(l) $x - 4 = -3y^2$		
(m) $-5x^2 - 5y^2 = -25$		
(n) $y - x^2 = 5$		
(o) $\frac{12}{xy} = 1$		
(p) $x^2 + 2y^2 - 4 = 0$		

2. For each equation,

   (i) state the type of graph it represents.
   (ii) determine the x-intercepts.
   (iii) determine the y-intercepts.
   (iv) sketch the graph, showing the x and y-intercepts and general shape. (A table of values is not required.)
   (v) state the domain and range of the relation.

(a) $\{(x, y) | x^2 + 4y^2 = 36\}$

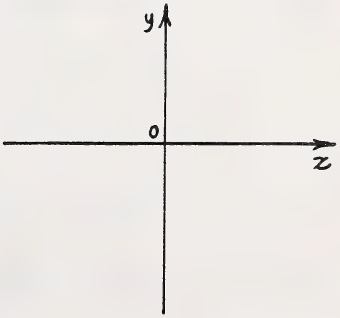
(i) This graph is a(n) ____________________.

(ii) x-intercepts	(iii) y-intercepts	(iv) Graph
Let $y = 0$.	Let $x = 0$.	

(v) The domain is ________________ and the range is ________________.

(b) $\{(x, y) | 2x^2 + 2y^2 = 32\}$

(i) The graph is a(n) ____________________.

(ii) x-intercepts	(iii) y-intercepts	(iv) Graph
		

(v) The domain is ________________ and the range is ________________.

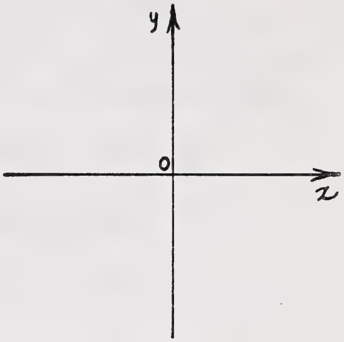
(c) $\{(x, y) | xy = -16\}$

(i) The graph is a(n) ____________________.

(ii) x-intercepts

(iii) y-intercepts

(iv) Graph



(v) The domain is ________________ and the range is ________________.

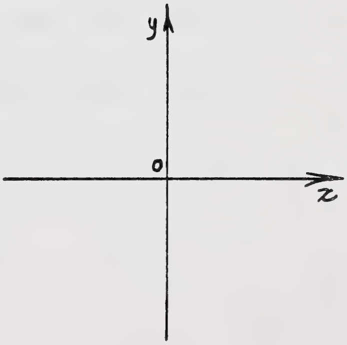
(d) $\{(x, y) | x = -y^2\}$

(i) The graph is a(n) ____________________.

(ii) x-intercepts

(iii) y-intercepts

(iv) Graph



(v) The domain is ________________ and the range is ________________.

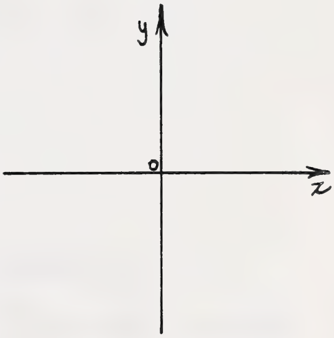
(e) $\{(x, y) | x^2 - 4y^2 = -16\}$

(i) The graph is a(n) ____________________.

(ii) x-intercepts

(iii) y-intercepts

(iv) Graph



(v) The domain is ________________ and the range is ________________.

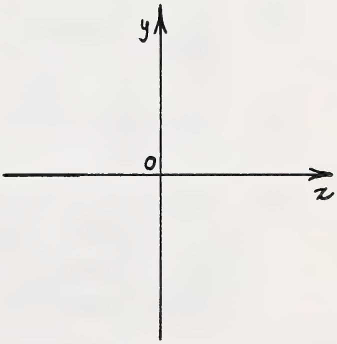
(f) $\{(x, y) | 4x^2 + y^2 = 16\}$

(i) The graph is a(n) ____________________.

(ii) x-intercepts

(iii) y-intercepts

(iv) Graph



(v) The domain is ________________ and the range is ________________.

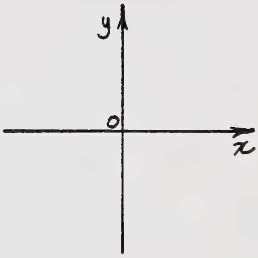
(g) $\{(x, y) \mid x^2 - y^2 = 25\}$

(i) The graph is a(n) ____________________.

(ii) x-intercepts

(iii) y-intercepts

(iv) Graph



(v) The domain is ____________________ and the range is ____________________.

3. Decide if each of the following statements is true or false.

True or False?

(a) The graph of $\{(x, y) \mid y = x^2 + 5\}$ is a parabola opening to the right. ____________________

(b) The domain of $\{(x, y) \mid x^2 + y^2 = 16\}$ is $\{x \mid -4 \leqq x \leqq 4\}$. ____________________

(c) The vertex of $\{(x, y) \mid x = y^2 - 5\}$ occurs at $(0, -5)$. ____________________

(d) The relation $\{(x, y) \mid x = 3y^2\}$ is a function. ____________________

(e) The graph of $\{(x, y) \mid x^2 - y^2 = 9\}$ is a circle. ____________________

(f) The graph of $\{(x, y) \mid x^2 - 3y^2 = -5\}$ is a hyperbola which cuts the y-axis in two places and never comes in contact with the x-axis. ____________________

(g) The graph of $\{(x, y) \mid xy = -4\}$ lies entirely in Quadrants II and IV. ____________________

(h) The graph of $\{(x, y) \mid xy = 36\}$ never cuts the x or y-axis. ____________________

4. For the parabola defined by each equation below, give the direction it opens, the coordinates of the vertex, and the equation of the axis of symmetry. Then state the domain and range of the relation.

(a) $\{(x, y) \mid y = x^2 - 2x - 5\}$

(i) The graph opens ____________________.

(ii)

$\frac{-b}{2a} =$	$\frac{4ac - b^2}{4a} =$

The vertex occurs at (___,___).

(iii) The equation of the axis of symmetry is __________.

(iv) The domain is ____________________________.

The range is ____________________________.

(b) $\{(x, y) \mid x = -5y^2 + 4y - 2\}$ (HINT: Review page 34.)

(i) The graph opens ____________________.

(ii)

$\frac{4ac - b^2}{4a} =$	$\frac{-b}{2a} =$

The vertex occurs at (___,___).

(iii) The equation of the axis of symmetry is __________.

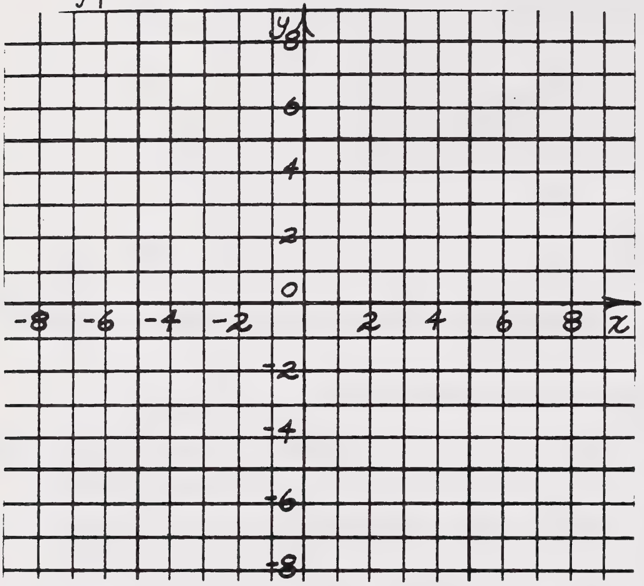
(iv) The domain is ____________________________.

The range is ____________________________.

5. Make a table of values and then graph the following relations. (Use values of x from the domain of the relation.) Room for rough work has been provided on the right.

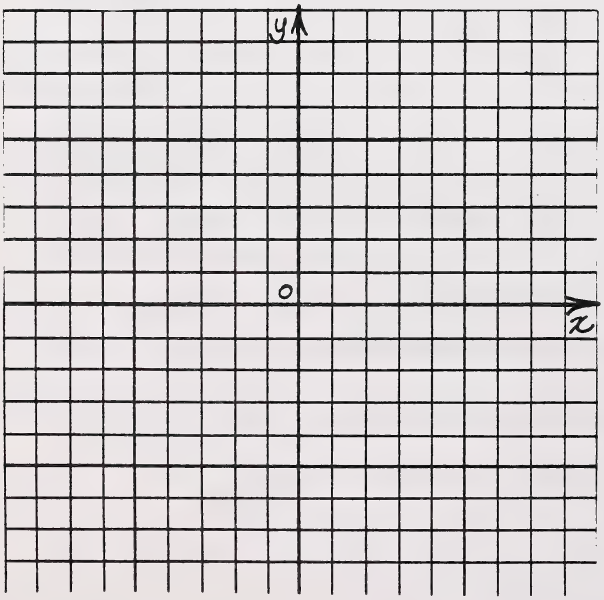
(a) $\{(x, y) \mid xy = -12\}$

x	
y	



(b) $\{(x, y) \mid x^2 + y^2 = 16\}$

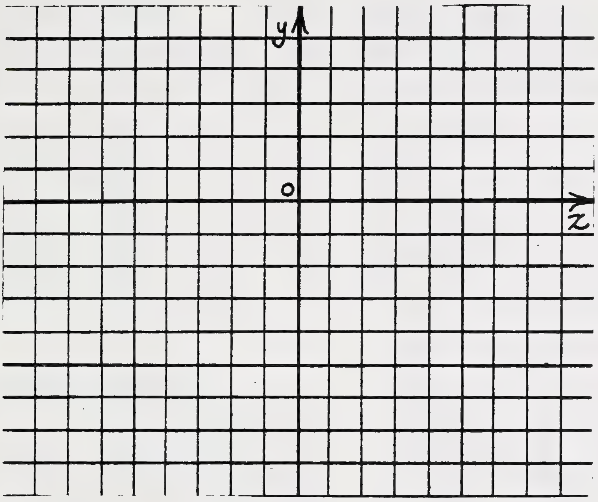
x	
y	



Rough Work

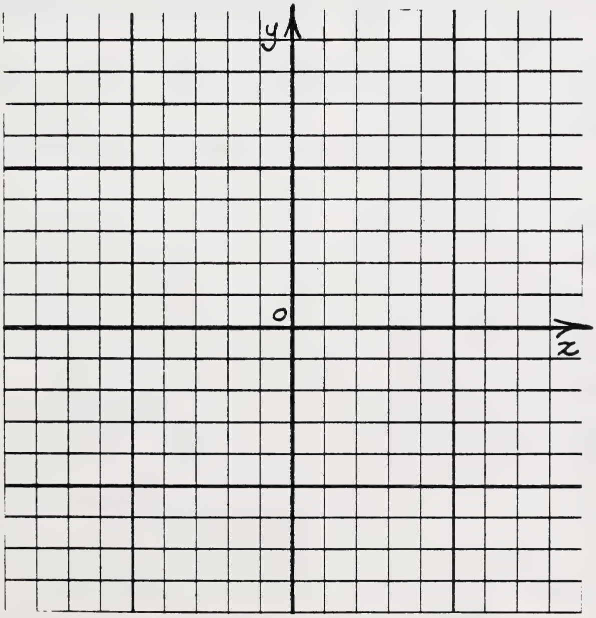
(c) $\{(x, y) \mid y = -x^2 + 3x\}$

x	
y	



(d) $\{(x, y) \mid x^2 - y^2 = 9\}$

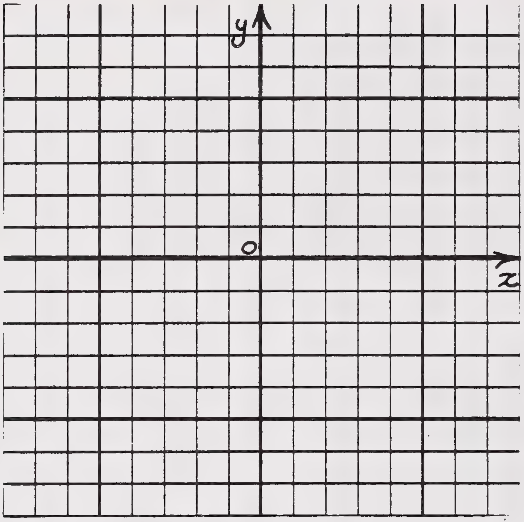
x	±3	±4	±6	±8
y				



Rough Work

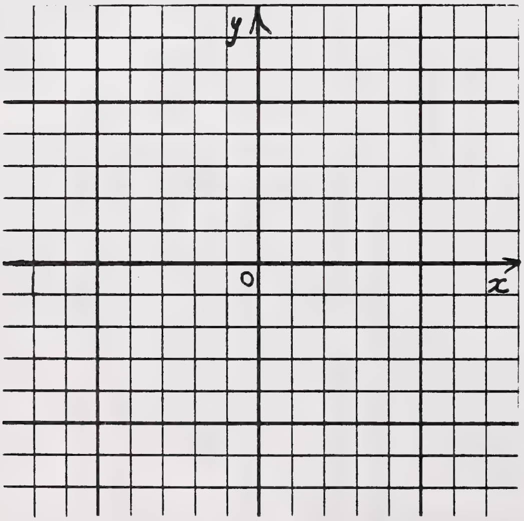
(e) $\{(x, y) \mid 3x^2 + y^2 = 27\}$

x	
y	



(f) $\{(x, y) \mid x^2 - 2y^2 = -8$

x	
y	



Rough Work

Key to Self-correcting Exercises in Lesson H

Exercise #1, page 3

1. (a) no (linear) (b) no (y^2 present) (c) yes ($y = x^2 + x$)
(d) no (inequality) (e) yes (f) no (y^2 present)
(g) yes ($y = x^2 - 5x$) (h) no (y^2 present)

2. (a) $\{(x,y) | y = x^2 + 3x\}$ a = 1, b = 3, c = 0
(b) $\{(x,y) | y = -2x^2 + 4x - 8\}$ a = -2, b = 4, c = -8
(c) $\{(x,y) | y = x^2\}$ a = 1, b = 0, c = 0
(d) $\{(x,y) | y = x^2 - 3\}$ a = 1, b = 0, c = -3
(e) $\{(x,y) | y = 2x^2 - 5x - 9\}$ a = 2, b = -5, c = -9
(f) $\{(x,y) | y = 2x^2 + 4x + 1\}$ a = 2, b = 4, c = 1

Exercise #2, page 6

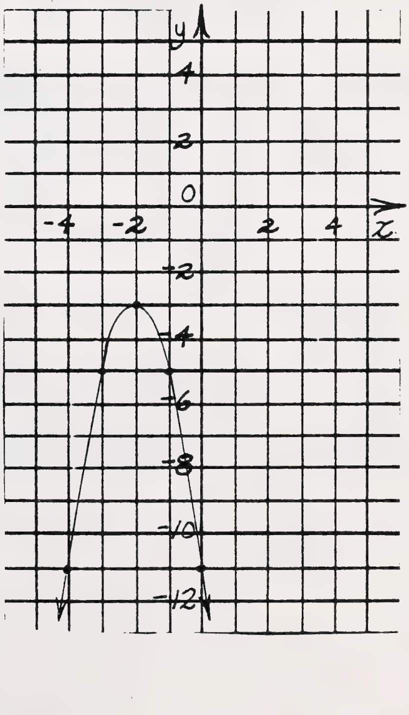
1.

	(a)	(b)
(i)	downward	upward
(ii)	$x = 0$ (y-axis)	$x = -2$
(iii)	(0,0)	(-2, -4)
(iv)	R	R
(v)	$\{y \mid y \leqq 0\}$	$\{y \mid y \geqq -4\}$
(vi)	0	-4 and 0
(vii)	0	0

2.

x	-4	-3	-2	-1	0
y	-11	-5	-3	-5	-11

Equation of axis of symmetry is
$x = -2$.
Coordinates of vertex are (-2, -3).



Exercise #3, page 10

1. (a) $y = 3(x - 0)^2 + (-7)$
$a = 3,\ k = 0,\ p = -7$

(b) $y = 5(x - 1)^2 + 0$
$a = 5,\ k = 1,\ p = 0$

(c) $y = -2[x - (-1)]^2 + (-4)$
$a = -2,\ k = -1,\ p = -4$

(d) $y = -1[x - (-3)]^2 + 0$
$a = -1,\ k = -3,\ p = 0$

(e) $y = 4(x - 0)^2 + 0$
$a = 4,\ k = 0,\ p = 0$

(f) $y = 1(x - 0)^2 + 1$
$a = 1,\ k = 0,\ p = 1$

2. (a) $x = 1$; $(1, 3)$
(b) $x = -3$; $(-3, -7)$
(c) $x = 0$; $(0, 0)$
(d) $x = 1$; $(1, 0)$
(e) $x = -4$; $(-4, -2)$
(f) $x = 0$; $(0, -5)$
(g) $x = -7$; $(-7, -1)$
(h) $x = 0$; $(0, -5)$

3. (a) up; min. (since $a = 2$)
(b) down; max. (since $a = -1$)
(c) down; max. (since $a = -3$)
(d) up; min. (since $a = 1$)
(e) down; max. (since $a = -2$)
(f) up; min. (since $a = 3$)

4. (a) "a" is negative.
$p = 3$
Range is $\{y \mid y \leqq 3\}$.

(b) "a" is positive.
$p = 0$
Range is $\{y \mid y \geqq 0\}$.

(c) "a" is positive.
$p = 7$
Range is $\{y \mid y \geqq 7\}$.

(d) "a" is negative.
$p = -4$
Range is $\{y \mid y \leqq -4\}$.

5.

(a) $y = -2(x - 3)^2 + 8$	$x = 3$	$(3, 8)$	$\{y \mid y \leqq 8\}$
(b) $y = (x + 3)^2 + 1$	$x = -3$	$(-3, 1)$	$\{y \mid y \geqq 1\}$
(c) $y = -(x - 4)^2$	$x = 4$	$(4, 0)$	$\{y \mid y \leqq 0\}$
(d) $y = 3(x + 1)^2 - 2$	$x = -1$	$(-1, -2)$	$\{y \mid y \geqq -2\}$
(e) $y = 4x^2$	$x = 0$	$(0, 0)$	$\{y \mid y \geqq 0\}$
(f) $y = -3x^2 + 5$	$x = 0$	$(0, 5)$	$\{y \mid y \leqq 5\}$
(g) $y = 7(x + 6)^2 - 1$	$x = -6$	$(-6, -1)$	$\{y \mid y \geqq -1\}$
(h) $y = -3(x - 5)^2 + 3$	$x = 5$	$(5, 3)$	$\{y \mid y \leqq 3\}$

Exercise #4, page 17

1.

(a) $y = x^2 - 3$	$y = 1(x - 0)^2 - 3$	1	0	-3	$x = 0$	(0, -3)
(b) $y = -5x^2$	$y = -5(x - 0)^2 + 0$	-5	0	0	$x = 0$	(0, 0)
(c) $y = \frac{-1}{2}x^2 + \frac{1}{4}$	$y = \frac{-1}{2}(x - 0)^2 + \frac{1}{4}$	$\frac{-1}{2}$	0	$\frac{1}{4}$	$x = 0$	$(0, \frac{1}{4})$
(d) $y = 2x^2$	$y = 2(x - 0)^2 + 0$	2	0	0	$x = 0$	(0, 0)
(e) $y = 3x^2 + 1$	$y = 3(x - 0)^2 + 1$	3	0	1	$x = 0$	(0, 1)

2. (a) $\left(\frac{1}{2} \times 7\right)^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$

(b) $\left(\frac{1}{2} \times \frac{-1}{2}\right)^2 = \left(\frac{-1}{4}\right)^2 = \frac{1}{16}$

(c) $\left(\frac{1}{2} \times 4\right)^2 = (2)^2 = 4$

(d) $\left(\frac{1}{2} \times -2\right)^2 = (-1)^2 = 1$

(e) $\left(\frac{1}{2} \times \frac{-3}{4}\right)^2 = \left(\frac{-3}{8}\right)^2 = \frac{9}{64}$

(f) $\left(\frac{1}{2} \times \frac{2}{3}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

(g) $\left(\frac{1}{2} \times -9\right)^2 = \left(\frac{-9}{2}\right)^2 = \frac{81}{4}$

(h) $\left(\frac{1}{2} \times -1\right)^2 = \left(\frac{-1}{2}\right)^2 = \frac{1}{4}$

3. (a)

$$y = x^2 + 2x + 3$$

$$y = (x^2 + 2x) + 3$$

$$y + \overset{\text{ADD 1}}{1} = (x^2 + 2x + \overset{\leftarrow \text{ADD 1.}}{1}) + 3$$

$$y + 1 = (x + 1)^2 + 3$$

$$y = (x + 1)^2 + 3 - 1$$

$$y = (x + 1)^2 + 2$$

Graph opens upward.
Axis of symmetry is $x = -1$.
Vertex occurs at $(-1, 2)$.
Range is $\{y | y \geqq 2\}$.

(b)

$$y = 2x^2 + 3x$$

$$y = (2x^2 + 3x)$$

$$y = 2\left(\frac{2x^2}{2} + \frac{3x}{2}\right)$$

$$y = 2\left(x^2 + \frac{3}{2}x\right)$$

$$y + \overset{\text{ADD } 2(9/16)}{\boxed{2\left(\frac{9}{16}\right)}} = 2\left(x^2 + \frac{3}{2}x + \overset{\text{ADD } 2(9/16).}{\frac{9}{16}}\right)$$

$$y + \frac{9}{8} = 2\left(x + \frac{3}{4}\right)^2$$

$$y = 2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8}$$

Graph opens upward.

Axis of symmetry is $x = \frac{-3}{4}$.

Vertex occurs at $\left(\frac{-3}{4}, -1\frac{1}{8}\right)$.

Range is $\{y | y \geqq -1\frac{1}{8}\}$.

Exercise #4, continued

2. (c)

$$y = -3x^2 + 15x - 5$$
$$y = (-3x^2 + 15x) - 5$$
$$y = -3\left(\frac{-3x^2}{-3} + \frac{15x}{-3}\right) - 5$$
$$y = -3(x^2 - 5x) - 5$$
$$y + \overset{\text{ADD } (-3)(25/4)}{\boxed{(-3)\frac{25}{4}}} = -3\left(x^2 - 5x + \overset{\text{ADD } (-3)(25/4).}{\frac{25}{4}}\right) - 5$$
$$y - \frac{75}{4} = -3\left(x - \frac{5}{2}\right)^2 - 5$$
$$y = -3\left(x - \frac{5}{2}\right)^2 - 5 + \frac{75}{4}$$
$$y = -3\left(x - \frac{5}{2}\right)^2 + \frac{55}{4}$$

Graph opens downward.

Axis of symmetry is $x = 2\frac{1}{2}$.

Vertex occurs at $(2\frac{1}{2}, 13\frac{3}{4})$.

Range is $\{y \mid y \leqq 13\frac{3}{4}\}$.

(d)

$$y = -x^2 + 2x - 2$$
$$y = (-x^2 + 2x) - 2$$
$$y = -1\left(\frac{-x^2}{-1} + \frac{2x}{-1}\right) - 2$$
$$y = -1(x^2 - 2x) - 2$$
$$y + \overbrace{(-1)(1)}^{\text{ADD } (-1)(1).} = -1(x^2 - 2x + \overset{\text{ADD } (-1)(1).}{1}) - 2$$
$$y - 1 = -(x - 1)^2 - 2$$
$$y = -(x - 1)^2 - 2 + 1$$
$$y = -(x - 1)^2 - 1$$

Graph opens downward.
Axis of symmetry is $x = 1$.
Vertex occurs at $(1, -1)$.
Range is $\{y \mid y \leqq -1\}$.

Exercise #5, page 21

1.

(a) $y = 3x^2 - 7$	3	0	-7	$\frac{-0}{2(3)}$ $= 0$	$\frac{4(3)(-7) - (0)^2}{4(3)}$ $= \frac{-84}{12}$ $= -7$
(b) $y = -2x^2 + 4x$	-2	4	0	$\frac{-4}{2(-2)}$ $= \frac{-4}{-4}$ $= 1$	$\frac{4(-2)(0) - (4)^2}{4(-2)}$ $= \frac{0 - 16}{-8}$ $= 2$
(c) $y = 5x^2 - 8x + 1$	5	-8	1	$\frac{-(-8)}{2(5)}$ $= \frac{4}{5}$	$\frac{4(5)(1) - (-8)^2}{4(5)}$ $= \frac{20 - 64}{20}$ $= \frac{-11}{5}$ or $-2\frac{1}{5}$

Exercise #5, continued

2. (a) $y = x^2 - x + 1$

$a = 1,\ b = -1,\ c = 1$

$$\frac{-b}{2a} = \frac{-(-1)}{2(1)} = \frac{1}{2}$$

Axis of symmetry is $x = \frac{1}{2}$.

$$\frac{4ac - b^2}{4a} = \frac{4(1)(1) - (-1)^2}{4(1)}$$

$$= \frac{4 - 1}{4} = \frac{3}{4}$$

Coordinates of vertex are $\left(\frac{1}{2}, \frac{3}{4}\right)$.

Range is $\{y \mid y \geqq \frac{3}{4}\}$.

(b) $y = -4x^2 + 8x + 5$

$a = -4,\ b = 8,\ c = 5$

$$\frac{-b}{2a} = \frac{-8}{2(-4)} = 1$$

Axis of symmetry is $x = 1$.

$$\frac{4ac - b^2}{4a} = \frac{4(-4)(5) - (8)^2}{4(-4)}$$

$$= \frac{-80 - 64}{-16} = 9$$

Coordinates of vertex are $(1, 9)$.

Range is $\{y \mid y \leqq 9\}$

Exercise #6, page 25

1. (a) Let $x = 0$.

$y = (0)^2 + (0) - 6$

$y = -6$

y-intercept is -6.

Let $y = 0$.

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

If $x + 3 = 0$ If $x - 2 = 0$

$x = -3$ $x = 2$

x-intercepts are -3 and 2. The graph cuts the x-axis twice.

(b) Let $x = 0$.

$y = 2(0)^2 + 12(0) + 18$

$y = 18$

y-intercept is 18.

Let $y = 0$.

$2x^2 + 12x + 18 = 0$

$x^2 + 6x + 9 = 0$

$(x + 3)(x + 3) = 0$

If $x + 3 = 0$

$x = -3$

x-intercept is -3. The graph comes in contact with the x-axis once.

(c) Let $x = 0$.

$y = (0)^2 - (0) + 1$

$y = 1$

y-intercept is 1.

Let $y = 0$.

$x^2 - x + 1 = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{-3}}{2} \quad \begin{cases}\text{not} \\ \text{real}\end{cases}$$

There is no x-intercept. The graph never comes in contact with the x-axis.

(d) Let $x = 0$.

$y = (0)^2 + 5(0) + 1$

$y = 1$

y-intercept is 1.

Let $y = 0$.

$x^2 + 5x + 1 = 0$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{21}}{2}$$

$$x = \frac{-5 \pm 4.583}{2}$$

$x = -4.8$ or -0.2

The x-intercepts are -4.8 and -0.2. The graph cuts the x-axis twice.

Exercise #7, page 34

1. (a) (i) right

(ii) $y = 0$ (x-axis)

(iii) $(-2, 0)$

(iv) $\{x \mid x \geqq -2\}$

(v) R

(vi) -2

(vii) 4 and -4

(b) (i) left

(ii) $y = 1$

(iii) $(4, 1)$

(iv) $\{x \mid x \leqq 4\}$

(v) R

(vi) 3

(vii) 3 and -1

2. (a) $\{(x, y) \mid x = 3y^2 - 5\}$

$a = 3,\ b = 0,\ c = -5$

$$\frac{4ac - b^2}{4a} = \frac{4(3)(-5) - (0)^2}{4(3)} = \frac{-60}{12} = \boxed{-5}$$

$$\frac{-b}{2a} = \frac{-0}{2(3)} = \boxed{0}$$

Axis of symmetry is $y = 0$.

Coordinates of vertex are $(-5, 0)$.

Domain is $\{x \mid x \geqq -5\}$ since $a > 0$.

(b) $\{(x, y) \mid x = -y^2 + 5y - 3\}$

$a = -1,\ b = 5,\ c = -3$

$$\frac{4ac - b^2}{4a} = \frac{4(-1)(-3) - 5^2}{4(-1)} = \frac{-13}{-4} = \frac{13}{4} \text{ or } \boxed{3\tfrac{1}{4}}$$

$$\frac{-b}{2a} = \frac{-5}{2(-1)} = \frac{5}{2} \text{ or } \boxed{2\tfrac{1}{2}}$$

Axis of symmetry is $y = 2\frac{1}{2}$.

Coordinates of vertex are $(3\frac{1}{4}, 2\frac{1}{2})$.

Domain is $\{x \mid x \leqq 3\frac{1}{4}\}$ since $a < 0$.

Exercise #8, page 37

1. (a) $x^2 + y^2 = 8$; $\sqrt{8} \doteq 2.8$

(b) $x^2 + y^2 = 49$; $\sqrt{49} = 7$

(c) $x^2 + y^2 = \frac{1}{4}$; $\sqrt{\frac{1}{4}} = \frac{1}{2}$

(d) $x^2 + y^2 = 5$; $\sqrt{5} \doteq 2.2$

(e) $x^2 + y^2 = 9$; $\sqrt{9} = 3$

2. (a) 3

(b) $\{x \mid -3 \leqq x \leqq 3\}$

(c) $\{y \mid -3 \leqq y \leqq 3\}$

(d) 3 and -3

(e) 3 and -3

Exercise #9, page 39

1. (a) $-2y^2 = -5 + 3x^2$
Subtract $3x^2$ from both sides.
$-3x^2 - 2y^2 = -5$
Multiply every term by -1.
$\underline{3x^2 + 2y^2 = 5}$

(b) $x^2 + 2y^2 - 1 = 0$
Add 1 to both sides.
$\underline{x^2 + 2y^2 = 1}$

(c) $4y^2 = 3 - 7x^2$
Add $7x^2$ to both sides.
$\underline{7x^2 + 4y^2 = 3}$

(d) $5x^2 = 8 - 3y^2$
Add $3y^2$ to both sides.
$\underline{5x^2 + 3y^2 = 8}$

(e) $5x^2 - 7 = -4y^2$
Add $4y^2$ to both sides.
$5x^2 + 4y^2 - 7 = 0$
Add 7 to both sides.
$\underline{5x^2 + 4y^2 = 7}$

(f) $9 - 4x^2 = 3y^2$
Add $4x^2$ to both sides.
$9 = 4x^2 + 3y^2$
(or) $\underline{4x^2 + 3y^2 = 9}$

2. (a) $\{x \mid -4 \leqq x \leqq 4\}$ (b) $\{y \mid -8 \leqq y \leqq 8\}$ (c) -4 and 4

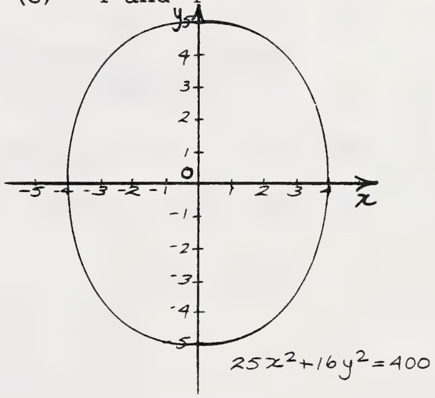
(d) -8 and 8

3. (a) Let y = 0.
$25x^2 + 16(0)^2 = 400$
$25x^2 = 400$
$x^2 = 16$
$\boxed{x = \pm 4}$

x-intercepts are ±4.

Let x = 0.
$25(0)^2 + 16y^2 = 400$
$16y^2 = 400$
$y^2 = 25$
$\boxed{y = \pm 5}$

y-intercepts are ±5.



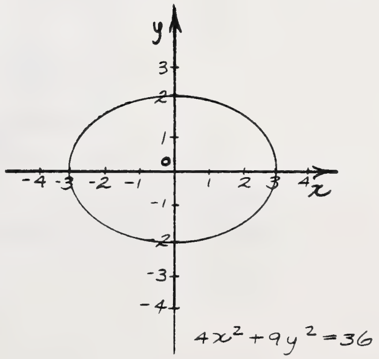
Mark these intercepts on the graph and then sketch the ellipse.

(b) Let y = 0.
$4x^2 + 9(0)^2 = 36$
$4x^2 = 36$
$x^2 = 9$
$\boxed{x = \pm 3}$

x-intercepts are ±3.

Let x = 0.
$4(0)^2 + 9y^2 = 36$
$9y^2 = 36$
$y^2 = 4$
$\boxed{y = \pm 2}$

y-intercepts are ±2.



Mark these intercepts on the graph and then sketch the ellipse.

Exercise #10, page 44

1. (a) $5y^2 = 3x^2 - 2$

Subtract $3x^2$ from both sides.

$-3x^2 + 5y^2 = -2$

Multiply every term by -1.

$\underline{3x^2 - 5y^2 = 2}$

The branches cut the x-axis since the equation is of the form $ax^2 - by^2 = c$.

(b) $2x^2 - 3y^2 + 1 = 0$

Subtract 1 from both sides.

$\underline{2x^2 - 3y^2 = -1}$

The branches cut the y-axis since the equation is of the form $ax^2 - by^2 = -c$.

(c) $x^2 = 3y^2 - 7$

Subtract $3y^2$ from both sides.

$\underline{x^2 - 3y^2 = -7}$

The branches cut the y-axis since the equation is of the form $ax^2 - by^2 = -c$

(d) $3y^2 - 2x^2 = -7$

Multiply every term by -1.

$\underline{2x^2 - 3y^2 = 7}$

The branches cut the x-axis since the equation is of the form $ax^2 - by^2 = c$.

(e) $y^2 + 5 = 2x^2$

Subtract y^2 from both sides.

$\underline{5 = 2x^2 - y^2}$

The branches cut the x-axis since the equation is of the form $ax^2 - by^2 = c$.

2. (a) $\{x \mid x \leqq -3 \text{ or } x \geqq 3\}$ (b) R (c) -3 and 3 (d) none

3. (a) $2x^2 - 5y^2 = 8$

Let $y = 0$.

$$\begin{aligned} 2x^2 - 5(0)^2 &= 8 \\ 2x^2 &= 8 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

x-intercepts are ±2.

Let $x = 0$.

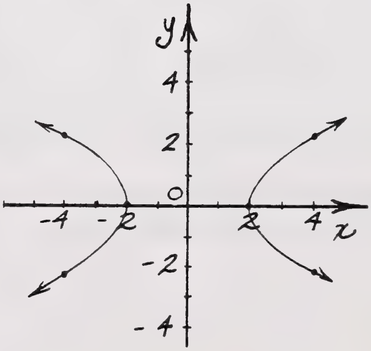
$$\begin{aligned} 2(0)^2 - 5y^2 &= 8 \\ -5y^2 &= 8 \\ y^2 &= \frac{-8}{5} \\ y &= \pm\sqrt{\frac{-8}{5}} \end{aligned}$$

y is not real.
There are no y-intercepts.

When $x = \pm 4$,

$$\begin{aligned} 2(\pm 4)^2 - 5y^2 &= 8 \\ 32 - 5y^2 &= 8 \\ -5y^2 &= -24 \\ y^2 &= 4.8 \\ y &= \pm\sqrt{4.8} \\ y &\doteq \pm 2.2 \end{aligned}$$

SEE $\sqrt{5}$ ON P. 57, LESSON B.



Exercise #10, continued

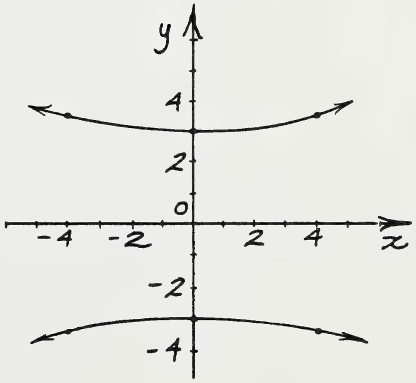
3. (b) $x^2 - 9y^2 = -81$

Let $y = 0$.
$x^2 - 9(0)^2 = -81$
$x^2 = -81$
$x = \pm\sqrt{-81}$
x is not real.
There are no x-intercepts.

Let $x = 0$.
$(0)^2 - 9y^2 = -81$
$-9y^2 = -81$
$y^2 = 9$
$y = \pm 3$
y-intercepts are ±3.

When $x = \pm 4$,
$(\pm 4)^2 - 9y^2 = -81$
$16 - 9y^2 = -81$
$-9y^2 = -97$
$y^2 \doteq 10.8$
$y \doteq \pm\sqrt{10.8}$
$y \doteq \pm 3.3$

SEE $\sqrt{11}$ ON P.57 LESSON B.



Exercise #11, page 47

1. (a) $xy = -4$; Quadrants II and IV (since k is negative).

(b) $xy = 7$; Quadrants I and III (since k is positive).

(c) $xy = 3$; Quadrants I and III (since k is positive).

(d) $xy = -5$; Quadrants II and IV (since k is negative).

(e) $xy = 6$; Quadrants I and III (since k is positive).

End of Lesson H

Lesson I

COMPLEX NUMBERS; EQUATION SYSTEMS

Topic One: The Complex Number System

A. Imaginary Numbers

Sets of numbers have been invented by mathematicians down through the centuries as the need arose. One of the basic problems confronting mathematicians of the past was the problem of solving a given algebraic equation. Whenever they encountered an equation whose solution could not be found in a given set of numbers, they invented a new set. This problem led in turn to the invention of the whole numbers, integers, rational numbers, and irrational numbers.

For instance, if we have only the set of natural numbers, $N = \{1, 2, 3, 4, \ldots\}$, then we can solve the equation $x + 3 = 5$. The solution set is $\{2\}$ and so $x \in N$, the set of natural numbers. The set N provides a solution set for all equations of the form

$$x + a = b,\ b > a,\ a, b \in N.$$

However, if $x + 3 = 3$, then $x \notin N$ because $\{0\}$ is the solution set. The solution for this equation is contained in the set of whole numbers, $W = \{0, 1, 2, 3, \ldots\}$. The set W provides a solution set for all equations of the form

$$x + a = b,\ b \geqq a,\ a, b \in W.$$

Again, if $x + 5 = 3$, then $x \notin W$ because $\{-2\}$ is the solution set. The solution for this equation is contained in the set of integers, $I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The set I provides a solution set for all equations of the form

$$x + a = b,\ a, b \in I.$$

If the equation is $3x = 5$, the set I will not do because $\left\{\frac{5}{3}\right\}$ is the solution set and $\frac{5}{3} \notin I$. The solution for this equation is contained in the set of rational numbers, $Q = \{\frac{a}{b} | a, b \in I,\ b \neq 0\}$.

The set Q provides a solution set for all equations of the form

$$bx = a,\ a, b \in Q, b \neq 0.$$

Even the set Q does not contain the solution to all the equations we may encounter. For if $x^2 = 2$, then $x \notin Q$ because $\{\sqrt{2}, -\sqrt{2}\}$ is the solution set, and x belongs to the set of irrational numbers. The union of the set of rationals with the set of irrationals is the set of real numbers, R. The set R provides a solution set for all equations of the form

$$x^2 = a, \text{ provided that } a \geqq 0,\ a \in R.$$

However, even with the set of real numbers, man was unable to solve all simple polynomial equations.

In Lesson H, you learned that some quadratic equations have no real roots. In general, the quadratic equation $ax^2 + bx + c = 0$ has no real roots if its discriminant, $b^2 - 4ac$, is less than zero. For example, the discriminant of the quadratic equation $x^2 + x + 4 = 0$ is $b^2 - 4ac$ or $(1)^2 - 4(1)(4) = 1 - 16 = -15$ which is negative. Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we find the roots of $x^2 + x + 4 = 0$ to be $\frac{-1 \pm \sqrt{-15}}{2}$. These roots are not real since $\sqrt{-15}$ is not a real number. (There is no real number whose square is -15.)

This suggests that the set of real numbers R should be extended so that every quadratic equation has a solution in this new enlarged system.

One of the simplest quadratic equations with no real roots is $x^2 + 1 = 0$. Note that:

$$\begin{aligned} \text{if } x^2 + 1 &= 0, \\ x^2 &= -1 \\ x &= \pm\sqrt{-1}. \end{aligned}$$

The roots of $x^2 + 1 = 0$ are not real since there is no real number whose square is -1.

For want of a better name, the number $\sqrt{-1}$ is called the "imaginary" and is designated by the letter, i. Thus, by definition,

$$\boxed{i = \sqrt{-1}.}$$

Since i and -i satisfy the equation $x^2 + 1 = 0$, it follows that:

$$\begin{aligned} (\pm i)^2 + 1 &= 0 \\ i^2 + 1 &= 0 \\ i^2 &= -1. \end{aligned}$$

Thus, the square of the imaginary number i is equal to the real number -1.

$$\boxed{i^2 = -1}$$

Any square root of a negative number may be written using the symbol i and can be called an "imaginary number." This can be done by writing the negative number that appears under the root sign as the product of a positive number and -1. Then the rule for multiplying radicals can be applied in reverse and $\sqrt{-1}$ can be replaced by i.

EXAMPLES:

$$\sqrt{-4} = \sqrt{4 \times \overset{\text{-1 IS A FACTOR.}}{-1}} = \sqrt{4} \times \sqrt{-1} = \overset{\sqrt{-1} = i}{\sqrt{4}i} = 2i$$

$$\sqrt{-9} = \sqrt{9 \times -1} = \sqrt{9} \times \sqrt{-1} = \sqrt{9}i = 3i$$

$$\sqrt{-11} = \sqrt{11 \times -1} = \sqrt{11} \times \sqrt{-1} = \sqrt{11}\underset{\text{NOTE THAT "i" IS \underline{NOT} UNDER THE ROOT SIGN.}}{i}$$

In general, for any positive number "a", $\sqrt{-a}$ is not real and can be written as the imaginary number $\sqrt{a}i$.

It is unfortunate that the word "imaginary" was originally attached to numbers involving $\sqrt{-1}$ or i, since at the present time we know that they are as "real" as any other numbers. As is true of all other sets of numbers, imaginary numbers are clearly defined and represented by symbols. They can be added, subtracted, multiplied, and divided according to set rules; they can be represented graphically and they have great practical value, especially in the field of electrical engineering. Yet the term "imaginary" has continued to be used to designate this set of numbers.

Self-correcting Exercise #1

Answers may be found on page 58 of this lesson.

1. Rewrite each of the following expressions, substituting i for $\sqrt{-1}$.

(a) $\sqrt{-2} =$

(b) $3 + \sqrt{-5} =$

(c) $\sqrt{-7} + \sqrt{-3} =$

(d) $\frac{\sqrt{-9}}{7} =$

(e) $\frac{-2 \pm \sqrt{-4}}{5} =$

(f) $\frac{\sqrt{-1}}{2} =$

2. Solve each equation and state the simplest set to which the solutions belong. (Choose from N, W, I, Q, and R.)

(a) $x + 5 = 8$

$x = 8 - 5$

$x = 3$

Simplest set: N

(b) $a + 3 = 1$

Simplest set: _____

(c) $x^2 + 7 = 11$

Simplest set: _____

(d) $y^2 = 6$

Simplest set: _____

(e) $x^2 - 6x = 0$

Simplest set: _____

(f) $3x - 5 = 0$

Simplest set: _____

B. The Set of Complex Numbers

Imaginary numbers along with real numbers make up the SET OF COMPLEX NUMBERS.

Any complex number may be written in the form a + bi where a and b are real numbers. For example, the following are all complex numbers.

$2 + 3i$ ($a = 2$ and $b = 3$)

$-5 + i$ ($a = -5$ and $b = 1$)

$0 + 2i$ ($a = 0$ and $b = 2$)

$\frac{-1}{2} + 0i$ ($a = \frac{-1}{2}$ and $b = 0$)

$-3 - i$ ($a = -3$ and $b = -1$)

For any complex number a + bi, if b = 0, the complex number a + 0i is the real number "a." For example, the complex number -5 + 0i is the same as the real number -5.

For any complex number a + bi, if a = 0, the complex number 0 + bi becomes the PURE IMAGINARY NUMBER bi. For example, 0 + 2i is the same as the pure imaginary number 2i.

The set of all real numbers and likewise the set of all pure imaginary numbers are now seen to be included in the more comprehensive set of all complex numbers. We will designate this set by the capital letter C.

$$C = \{a + bi \mid a \in R,\ b \in R\}.$$

Self-correcting Exercise #2

Answers may be found on page 58 of this lesson.

1. Classify each of the following complex numbers as being "real", "pure imaginary" or "neither."

(a) $3i$ ____________

(b) $-9 + 0i$ ____________

(c) $\sqrt{2}i$ ____________

(d) $\frac{5 + i}{2}$ ____________

(e) $\frac{0 - 3i}{4}$ ____________

(f) $-\pi$ ____________

2. For each complex number below, express it in the form a + bi, and then state the values of "a" and "b."

	Form a + bi	Value of a	Value of b
(a) $\frac{-4 + 2i}{4}$	$-1 + \frac{1}{2}i$	-1	$\frac{1}{2}$
(b) 3i			
(c) $\frac{1 + i}{3}$			
(d) $-7\frac{1}{2}$			
(e) $\frac{5 - 15i}{3}$			
(f) $\frac{-12 - 2i}{2}$			

C. Operations With Complex Numbers

Since the set of real numbers is a subset of the set of complex numbers, we should be able to operate with complex numbers in much the same way as we do real numbers. We just need to keep in mind that, by definition, $i^2 = -1$. Thus, whenever i^2 appears in an expression, it can be replaced by -1.

In the sections that follow, we are going to consider how to add, subtract, multiply and divide two complex numbers. We will define these operations in set C in such a way that the commutative, associative, and distributive properties of real numbers still hold. Also, we will draw a parallel between operations with complex numbers and operations with linear monomials and binomials in algebra.

I. Adding and Subtracting Complex Numbers

First, let us consider the sum or difference of pure imaginary numbers. These operations can be compared with the sum or difference of like terms in algebra. On pages 30 and 31, lesson C, you were shown how the distributive property can be applied in reverse to help us add or subtract like algebraic terms.

EXAMPLE:

$$\begin{aligned} &4x - 6x + 5x - 7x \\ &= (4 - 6 + 5 - 7)x \longleftarrow \text{Distributive Property (Think this step. Don't write it.)} \\ &= \underline{\underline{-4x}} \end{aligned}$$

In effect, we have added and subtracted the numerical coefficients and multiplied the result by the common literal coefficient.

This same method can be used to add or subtract pure imaginary numbers.

EXAMPLE:

$$\begin{aligned} &-2i + 9i + 4i - 7i \\ &= (-2 + 9 + 4 - 7)i \longleftarrow \text{Distributive Property (Think this step. Don't write it.} \\ &= \underline{\underline{4i}} \end{aligned}$$

In general, for any pure imaginary numbers bi and ci,

$$bi + ci = (b + c)i$$
$$bi - ci = (b - c)i$$

Fill in the blanks in the chart below in order to find the sums and differences.

	Think Step	Simplification
$-12i + 11i$	$(-12 + 11)i$	$-i$
$7i + i$	$(7 + 1)i$	________
$8i - 3i$	________	________
$5i + 4i - 7i$	________	________
$-8i + 2i - 3i$	________	________
$5i - 6i + i$	________	________
$3i - 5i + 6i - 8i$	________	________
$2i - i + 3i - 6i$	________	________

Now let us consider the sum of two complex numbers that have both a real and an imaginary part. Such a sum can be compared with the sum of linear binomials in algebra. On page 20, lesson D, you were told that the commutative and associative properties of addition allow us to change the order and manner of grouping terms in a sum so that like terms can be grouped together. Then the distributive property can be used to add the numerical coefficients of like terms.

EXAMPLE: $(3x - 5) + (-2x + 8)$
Group like terms together.
$= (3x - 2x) + (-5 + 8)$
Combine like terms.
$= \underline{\underline{x + 3}}$

This same method can be used to find the sum of any two complex numbers.

EXAMPLE: $(-3 + 5i) + (6 - 8i)$
Group the real parts, -3 and 6, together and the imaginary parts, 5i and -8i, together.
$= (-3 + 6) + (5i - 8i)$
Combine like terms.
$= \underline{\underline{3 - 3i}}$

In general, for any complex numbers $(a + bi)$ and $(c + di)$,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Find the following sums.

1. $(4 + 2i) + (-6 + 5i) = (4 - 6) + (2i + 5i) =$ ___ + ___i
2. $(6 + 2i) + (3 - 7i) = (\quad + \quad) + (2i - 7i) =$ ___ - ___i
3. $(-3 - 4i) + (2 + 7i) = (\quad + \quad) + (\quad + \quad) =$ ___ + ___i
4. $3i + (5 - 7i) = 5 + (3i - 7i) =$ ___ - ___i
5. $(-6 - 5i) + 8i =$
6. $-8 + (3 - 2i) =$
7. $(-2 + 8i) + (7 - 9i) =$

Finally, let us look at the difference of two complex numbers that have both a real and an imaginary part. Such a difference can be compared with the difference of linear binomials in algebra. On page 24, lesson D, you were told that in order to subtract one polynomial from another, you must add the additive inverse of the second polynomial to the first polynomial. (Recall that the additive inverse of a polynomial can be determined by taking the additive inverse of each of its terms.)

EXAMPLE: $(6x - 3y) - (-2x + 5y)$

Change the operation to addition and give the additive inverse of the second binomial. (The additive inverse of $-2x + 5y$ is $2x - 5y$.)

$$= (6x - 3y) \underset{\text{OPERATION IS CHANGED TO ADDITION.}}{+} \underbrace{(2x - 5y)}_{\text{THIS IS THE ADDITIVE INVERSE } -2x+5y.}$$

Add as before by grouping like terms.

$$= (6x + 2x) + (-3y - 5y)$$

$$= \underline{\underline{8x - 8y}}$$

This same method can be used to find the difference of any two complex numbers.

EXAMPLE: $(-6 - 7i) - (2 - 5i)$

$$= (-6 - 7i) \underset{\text{CHANGE OPERATION TO ADDITION.}}{+} \underbrace{(-2 + 5i)}_{\text{THIS IS THE ADDITIVE INVERSE OF } 2-5i.}$$

Proceed in addition by grouping the real parts together and the imaginary parts together.

$$= (-6 - 2) + (-7i + 5i)$$

$$= \underline{\underline{-8 - 2i}}$$

> In general, for any complex numbers $(a + bi)$ and $(c + di)$,
>
> $$(a + bi) - (c + di) = (a + bi) + (-c - di)$$

Find the following differences.

1. $(6 - 5i) - (3 + 2i) = (6 - 5i) + (-3 - 2i) = (6 - 3) + (-5i - 2i) =$ ______

2. $(-6 + 4i) - (-2 + 5i) = (-6 + 4i) + (2 - 5i) = (-6 + \quad) + (4i - \quad) =$ ______

3. $(3 + 2i) - (-4 - i) = (3 + 2i) + (\qquad) = (3 + \quad) + (2i + \quad) =$ ______

4. $6i - (4 + 8i) = 6i + (\qquad) = -4 + (\quad - \quad) =$ ______

5. $(3 - 4i) - (5 - 7i) = (3 - 4i) + (\qquad) =$ ______

Self-correcting Exercise #3

Answers may be found on page 59 of this lesson.

1. Simplify the following sums and differences.

(a) $4i + 7i =$ ______

(b) $2i - 6i =$ ______

(c) $-5i - 2i =$ ______

(d) $-3i + 9i =$ ______

(e) $8i - i + 3i =$ ______

(f) $2i - 6i - 3i =$ ______

(g) $(3 + 5i) - i$
=

(h) $(5 + 4i) + (6 + 2i)$
=

(i) $(3 + 2i) + (5 - 6i)$
=

(j) $(-6 + 4i) + (2 - 5i)$
=

(k) $5 - (2 + 4i)$
=

(l) $(3 + i) - (4 - i)$
=

(m) $(7 - 9i) - (-1 + 3i)$
=

(n) $(5 + 3i) - 8i + (2 + i)$
=

(o) $5 + (-8 + i) + 3i$
=

(p) $(2 + i) - (3 - i) - (7 + 2i)$
=

II. Multiplying Complex Numbers

First, let us consider the product of two pure imaginary numbers. Such a product can be compared with the product of monomials in algebra. On page 27, lesson D, you were told that two monomials can be multiplied by grouping numerical factors together and grouping literal factors with like bases together. The Product of Powers Property can be used to simplify the product of variable factors with like bases.

EXAMPLE:

$$-5x^2 \times 4x^3$$

Group the numerical factors -5 and 4 together.
Group the factors x^2 and x^3 (that have base x) together.

$$= (-5 \times 4)(x^2 \times x^3)$$

Now, multiply the numerical factors and use the Product of Powers Property to simplify the product of the powers with base x.

$$= -20 \times x^{2+3}$$

$$= \underline{\underline{-20x^5}}$$

This same method can be used to multiply pure imaginary numbers. Only one additional fact need be kept in mind. If the factor i^2 appears in the product, it can be replaced by -1.

EXAMPLE:

$$-6i \times 2i$$

$$= (-6 \times 2)(i \times i)$$

PRODUCT OF REAL NUMBERS → ← PRODUCT OF i- FACTORS

$$= -12 \times i^2$$

i^2 HAS BEEN REPLACED BY -1.

$$= -12 \times -1$$

$$= 12$$

In the example above, note that the product of the two pure imaginary numbers -6i and 2i is the real number, 12.

In general, for any pure imaginary numbers bi and ci,

$$bi \times ci = bci^2 = bc(-1) = -bc$$

THE RESULT IS A REAL NUMBER.

Find the following products. In each case, simplify the product by replacing i^2 by -1.

1. $-7i \times -2i = (-7 \times -2)(i \times i) = 14i^2 = 14(-1) = \underline{\qquad}$
2. $6i \times 3i = (6 \times \underline{\quad})(i \times i) =$
3. $3i \times i = (\underline{\quad} \times 1)(i \times i) =$
4. $2i \times -8i = (\quad \times \quad)(\quad \times \quad) =$
5. $-7i \times -i =$

Next, let us consider the product of a real or a pure imaginary number and a complex number with both a real and an imaginary part. Such a product can be compared with the product of a monomial and a binomial in algebra. On pages 27 and 28, lesson D, you were told that in order to multiply a polynomial by a monomial, you must multiply each term of the polynomial by the monomial and find the sum of these individual products.

EXAMPLE:

$$\begin{aligned} & 3a(5a - b) \\ = \; & (3a)(5a) + (3a)(-b) \longleftarrow \text{Distributive Property} \\ = \; & \underline{\underline{15a^2 - 3ab}} \end{aligned}$$

This same method can be used to multiply a complex number by a real number or a pure imaginary number.

EXAMPLE 1:

$$\begin{aligned} & \overset{\text{REAL NUMBER}\searrow}{-3}(2 + 5i) \\ = \; & (-3)(2) + (-3)(5i) \longleftarrow \text{Distributive Property} \\ = \; & \underline{\underline{-6 - 15i}} \end{aligned}$$

EXAMPLE 2:

$$\begin{aligned} & \overset{\text{PURE IMAGINARY NUMBER}\searrow}{3i}(5 - 3i) \\ = \; & (3i)(5) + (3i)(-3i) \longleftarrow \text{Distributive Property} \\ = \; & 15i - 9i^2 \\ = \; & 15i + 9 \qquad \longleftarrow i^2 \text{ has been replaced by } -1. \\ \text{(or)} \quad = \; & \underline{\underline{9 + 15i}} \qquad \longleftarrow \text{Form } a + bi \end{aligned}$$

In general,

(i) for any real number "a" and complex number $c + di$,

$$a(c + di) = ac + adi.$$

(ii) for any pure imaginary number "ai" and complex number $c + di$,

$$ai(c + di) = aci + adi^2 = aci - ad.$$

Find the following products. (Replace i^2 by -1.)

1. $3i(-5 + 7i) = (3i)(-5) + (3i)(7i) = \underline{\qquad} + 21i^2 = \underline{\qquad} - 21$

2. $-4(1 + i) = (-4)(\quad) + (-4)(\quad) = \underline{\qquad} - \underline{\qquad}i$

3. $-2i(-3 + 5i) = (-2i)(\quad) + (\quad)(\quad) =$

4. $\frac{1}{2}(4 - 6i) =$

5. $i(1 - i) =$

Finally, let us consider the product of two complex numbers that both have a real and an imaginary part. Such a product can be compared with the product of two linear binomials in algebra. On pages 28 and 29, lesson D, you were told that in order to multiply two binomials you must multiply each term of one binomial by each term of the other and add the four products.

EXAMPLE:

$$\begin{aligned} &(3x + 4)(3x - 5) \\ &= \underbrace{(3x)(3x) + 4(3x) + (3x)(-5) + (4)(-5)}_{\text{4 SEPARATE PRODUCTS}} \\ &= 9x^2 + \underbrace{12x - 15x}_{\text{LIKE TERMS}} - 20 \\ &= \underline{\underline{9x^2 - 3x - 20}} \end{aligned}$$

This same method can be used to multiply two complex numbers. Again, i^2 can be replaced by -1.

EXAMPLE:

$$\begin{aligned} &(-2 + 3i)(5 - 6i) \\ &= (-2)(5) + (3i)(5) + (-2)(-6i) + (3i)(-6i) \\ &= -10 + \underbrace{15i + 12i}_{\text{LIKE TERMS}} - 18i^2 \quad \nwarrow i^2 = -1 \\ &= -10 + 27i + 18 \\ &= \underline{\underline{8 + 27i}} \end{aligned}$$

In general, for any complex numbers $a + bi$ and $c + di$,

$$\begin{aligned} (a + bi)(c + di) &= ac + bci + adi + bdi^2 \\ &= ac + (bc + ad)i - bd \\ &= (ac - bd) + (bc + ad)i \end{aligned}$$

Find the following products. (Replace i^2 by -1.)

1. $(2 - 3i)(5 - 4i) = 10 - 15i - 8i + 12i^2 = 10 - 23i - 12 =$ ___ - ___i

2. $(1 + i)(2 + 3i) = 2 +$ ___i + ___i + ___i^2 = ____________

3. $(-3 + 5i)(-2 + 2i) = 6 -$ ___i - ___i + ___i^2 = ____________

4. $(-1 - 2i)(3 + i) =$ ___ $- 6i - i -$ ___i^2 = ____________

5. $(4 - 5i)(2 + i) =$ ____________

In some cases when you find the product of two complex numbers of the form $(a + bi)$ and $(c + di)$, the product is a real number.

EXAMPLES:

$$(2 + 3i)(2 - 3i) = 4 + 6i - 6i - 9i^2 = 4 + 9 = 13$$
$$(-3 + 5i)(-3 - 5i) = 9 - 15i + 15i - 25i^2 = 9 + 25 = 34$$
$$(4 - i)(4 + i) = 16 - 4i + 4i - i^2 = 16 + 1 = 17$$

REAL NUMBERS

In each of the products above, note that the two factors of the product have a special feature. They differ only in the sign of their imaginary coefficients. Pairs of complex numbers that have the same real parts and imaginary parts that are additive inverses of each other are called COMPLEX CONJUGATES.

DEFINITION - COMPLEX CONJUGATES

For every complex number z, if $z = a + bi$, then the conjugate of z, denoted by $\bar{z}$, is $a - bi$.

The product of complex conjugates is always a real number. This product can be found most easily by squaring the first term, squaring the second term, and subtracting the two squares. Review pages 28 and 29, lesson D, where the product of two binomials which results in a binomial is discussed.

EXAMPLE:

$$(4 + 7i)(4 - 7i) \longleftarrow \text{Product of complex conjugates}$$
$$= \underset{\text{SQUARE OF FIRST TERM}}{(4)^2} \underset{\text{SUBTRACT}}{-} \underset{\text{SQUARE OF SECOND TERM}}{(7i)^2}$$
$$= 16 - 49i^2$$
$$= 16 + 49$$
$$= \underline{\underline{65}}$$

In general, for any complex number $a + bi$ and its conjugate $a - bi$,

$$(a + bi)(a - bi) = a^2 - b^2i^2$$
$$= a^2 + b^2$$

Find the following products.

1. $(5 + 4i)(5 - 4i) = 25 - 16i^2 = 25 + \_\_\_ = \_\_\_$

2. $(-6 + i)(-6 - i) = 36 - i^2 = \_\_\_ + \_\_\_ = \_\_\_$

3. $(2 - 8i)(2 + 8i) = \_\_ - \_\_ = \_\_\_\_\_\_\_\_\_\_$

4. $(-2 - 5i)(-2 + 5i) = \_\_ - \_\_ = \_\_\_\_\_\_\_\_\_\_$

5. $(1 + i)(1 - i) =$ ________________

Self-correcting Exercise #4

Answers may be found on page 60 of this lesson.

1. State the conjugate of each of the following complex numbers.

(a) $3 - i$ ________ (b) $-2 + 4i$ ________ (c) $5 + 2i$ ________

(d) $-3 - 4i$ ________ (e) $1 + i$ ________ (f) $-4 - i$ ________

2. Find the following products.

(a) $-2i \times 6i =$ ________________

(b) $-5i \times -i =$ ________________

(c) $-3(4 + 2i) =$ ________________

(d) $2(5 - i) =$ ________________

(e) $i(3 + i) =$ ________________

(f) $-5i(-2 + i) =$ ________________

(g) $(4 + 2i)(4 - 2i) =$ ________________________

(h) $(7 + 6i)(8 - 3i) =$ ________________________

(i) $(6 - 5i)(-4 + 3i) =$ ________________________

(j) $(5 - 6i)(2 - 3i) =$ ________________________

(k) $(3 + 7i)(3 - 7i) =$ ________________________

III. Dividing Complex Numbers

First, let us consider the quotient of two pure imaginary numbers. Such a quotient can be compared with the quotient of monomials in algebra. On page 31, lesson D, you were told that two monomials can be divided by finding the quotient of the numerical coefficients and multiplying this result by the quotient of variable factors with like bases.

EXAMPLE:

$$\frac{-8x^2}{4x}$$

Group the numerical factors -8 and 4 together.
Group the factors x^2 and x (with base x) together.

$$= \frac{-8}{4} \times \frac{x^2}{x}$$

Now, divide the numerical factors and use the Quotient of Powers Property to simplify the quotient of powers with base x.

$$= -2 \times x^{2-1}$$

$$= \underline{\underline{-2x}}$$

This same method can be used to divide pure imaginary numbers.

EXAMPLE:

$$= \frac{-10i}{-5i}$$

$$= \underbrace{\frac{-10}{-5}}_{\text{QUOTIENT OF REAL NUMBERS}} \times \underbrace{\frac{i}{i}}_{\text{QUOTIENT OF } i\text{-FACTORS}}$$

$$= 2 \times 1$$

$$= \underline{\underline{2}}$$

> In general, for any pure imaginary numbers bi and ci,
>
> $$\frac{bi}{ci} = \frac{b}{c}$$

Find the following quotients.

1. $\frac{2i}{i} = \frac{2}{1} \times \frac{i}{i} =$

2. $\frac{-15i}{3i} = \frac{-15}{3} \times \underline{\qquad} =$

3. $\frac{\frac{1}{2}i}{4i} =$

4. $\dfrac{-12i}{-4i} =$

5. $\dfrac{i}{5i} =$

Next, let us consider the quotient of a real number and a pure imaginary number. In simplifying such quotients, we must use a process that is similar to the process of "rationalizing the denominator" that we used when finding the quotient of radicals. You will recall that when we divide two radicals in which the radicand in the denominator is not a factor of the radicand in the numerator, we multiply both terms in the quotient by a radical that is identical to the divisor. This does not alter the quotient since it is the same as multiplying by one. By doing this, we make the divisor a rational number. (Review page 21, lesson C.)

EXAMPLE: $\dfrac{\sqrt{5}}{\sqrt{3}}$

This quotient can be simplified by multiplying through by $\dfrac{\sqrt{3}}{\sqrt{3}}$. (This is the same as multiplying by 1.)

$$= \frac{\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{15}}{3} \longleftarrow \text{The denominator is now a rational number.}$$

When working with the quotient of a real number and a pure imaginary number, we can make the divisor a real number (rather than a complex number) by multiplying both terms of the quotient by i.

EXAMPLE: $\dfrac{-3}{2i}$

Multiply through by $\dfrac{i}{i}$. (This is the same as multiplying by 1.)

$$= \frac{-3}{2i} \times \frac{i}{i}$$

$$= \frac{-3i}{2i^2}$$

$$= \frac{-3i}{2(-1)}$$

$$= \frac{3i}{2} \longleftarrow \text{The denominator is now a real number.}$$

In general, for any real number "a" and complex number bi,

$$\frac{a}{bi} = \frac{a}{bi} \times \frac{i}{i} = \frac{ai}{bi^2} = \frac{-ai}{b}$$

Simplify the following quotients. (Replace i^2 by -1.)

1. $\frac{5}{i} = \frac{5}{i} \times \frac{i}{i} = \frac{5i}{i^2} = \frac{5i}{-1} =$

2. $\frac{-1}{2i} = \frac{-1}{2i} \times \frac{i}{i} =$

3. $\frac{4}{5i} =$

4. $\frac{-3}{7i} =$

Finally, let us consider the quotient of two complex numbers that both have a real and an imaginary part. Such a quotient can also be simplified by rationalizing the denominator. Both terms in the quotient must be multiplied by the conjugate of the denominator. The denominator will then be a real number. (You will recall that the product of a complex number and its conjugate is always a real number.)

EXAMPLE: $\frac{5 + 2i}{3 - 5i}$

The conjugate of the denominator is $3 + 5i$.

Multiply through by $\frac{3 + 5i}{3 + 5i}$.

$$= \frac{5 + 2i}{3 - 5i} \times \frac{3 + 5i}{3 + 5i}$$

$$= \frac{15 + 6i + 25i + 10i^2}{9 - 25i^2}$$

$$= \frac{15 + 31i - 10}{9 + 25}$$

$$= \frac{5 + 31i}{34} \longleftarrow \text{Denominator is now a real number.}$$

$$= \frac{5}{34} + \frac{31}{34}i \longleftarrow \text{Form } a + bi$$

In general, for any complex numbers a + bi and c + di,

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$

$$= \frac{ac + (bc - ad)i + bd}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

Simplify the following quotients. (Put them in the form a + bi.)

1. $$\frac{3 + 2i}{-6 + 5i} = \frac{3 + 2i}{-6 + 5i} \times \frac{-6 - 5i}{-6 - 5i}$$

$$= \frac{-18 - 12i - 15i - 10i^2}{36 - 25i^2}$$

$$= \frac{-18 - 27i + \underline{\qquad}}{36 + \underline{\qquad}}$$

$$= \frac{\underline{\qquad}}{61}$$

$$= \frac{\underline{\quad}}{61} - \frac{\underline{\quad}}{61}i$$

2. $$\frac{2 + 3i}{5 + 2i} = \frac{2 + 3i}{5 + 2i} \times \frac{\underline{\qquad}}{5 - 2i}$$

$$=$$

Self-correcting Exercise #5

Answers may be found on page 60 of this lesson.

1. Find the following quotients. (Put answers for parts (b), (c) and (e) in the form a + bi.)

(a) $\frac{-20i}{-4i} =$

(b) $\frac{3}{2 - 5i} =$

(c) $\frac{2 + i}{-1 - i} =$

(d) $\frac{-5}{3i} =$

(e) $\frac{2 + i}{1 + 3i} =$

D. Powers of i

Earlier in this lesson, you learned that $i^2 = -1$. We can use this information to help us simplify other positive integral powers of i. We can write such a power as a product, one of whose factors is i^2 raised to a power. (The other factor is i or 1.) Since $i^2 = -1$, we can substitute -1 for i^2 and thus write the power in simpler form. Remember that $(-1)^n$ equals -1 if n is odd and equals +1 if n is even.

EXAMPLES:

$$i^3 = \boxed{(i^2)^1} \times \boxed{i} = -1 \times i = -i$$

$$i^4 = \boxed{(i^2)^2} \times \boxed{1} = (-1)^2 \times 1 = 1$$

$$i^5 = \boxed{(i^2)^2} \times \boxed{i} = (-1)^2 \times i = i$$

$$i^6 = \boxed{(i^2)^3} \times \boxed{1} = (-1)^3 \times 1 = -1$$

OTHER FACTOR IS i OR 1.

i^2 RAISED TO A POWER

Similarly,

$i^9 = (i^2)^4 \times i = (-1)^4 \times i = 1 \times i =$ ____

$i^{12} = (i^2)^6 \times 1 =$ ____________________

$i^{23} = (i^2) \times$ __ $=$ ____________________

$i^{14} =$ ____________________________

Whenever we simplify any positive integral power of i, the result will equal -i, i, 1, or -1.

Self-correcting Exercise #6

Answers may be found on page 61 of this lesson.

1. Simplify the following expressions by evaluating powers of i.

(a) $i^{15} =$

(b) $i^8 =$

(c) $2i - 3i^3 =$

(d) $6i^4 - 2i^5 =$

(e) $i \times 3i \times -5i =$

E. Quadratic Equations With Complex Roots

With the invention of complex numbers, mathematicians ended their quest for the roots of polynomial equations. No matter what polynomial equation is written, the roots will be complex numbers.

Thus, the roots of the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ will always be complex numbers.

- If $b^2 - 4ac \geqq 0$, the roots are real.
- If $b^2 - 4ac < 0$, the roots are imaginary and can be expressed in the form $a + bi$.

In the two examples below, note how the imaginary roots can be written using i for $\sqrt{-1}$.

EXAMPLE 1:

$$\begin{aligned} x^2 + 5 &= 0 \\ x^2 &= -5 \\ x &= \pm\sqrt{-5} \\ x &= \pm\sqrt{5}i \end{aligned}$$

EXAMPLE 2:

$$\begin{aligned} x^2 + x + 3 &= 0 \\ x &= \frac{-b \pm\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm\sqrt{(1)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{-1 \pm\sqrt{1 - 12}}{2} \\ &= \frac{-1 \pm\sqrt{-11}}{2} \\ &= \frac{-1 \pm\sqrt{11}i}{2} \\ &= \frac{-1}{2} \pm \frac{\sqrt{11}}{2}i \longleftarrow \text{Form } a + bi \end{aligned}$$

Self-correcting Exercise #7

Answers may be found on page 61 of this lesson.

1. Find the roots of the following quadratic equations. (Where applicable, replace $\sqrt{-1}$ by i.)

(a) $x^2 - 3 = 0$ (b) $x^2 + 3 = 0$

(c) $x^2 + x - 3 = 0$ (d) $x^2 - 3x + 4 = 0$

(e) $x^2 - 6x + 13 = 0$ (f) $2x^2 - 5x - 3 = 0$

F. Representing Complex Numbers Graphically

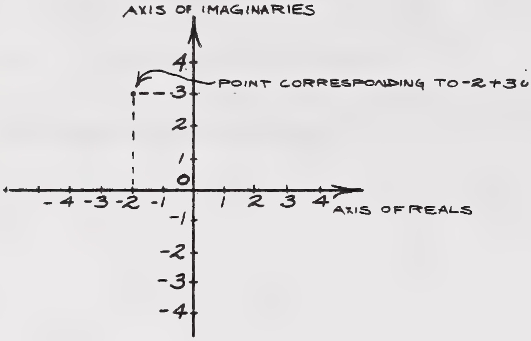
On page 49, lesson B, you learned that there is a one-to-one correspondence between the set of real numbers and the set of points in a solid number line. This means that every real number can be represented by a point in the number line and conversely, every point in the number line can be associated with a real number.

When we wish to represent complex numbers graphically, we must move into two dimensions. The complex numbers correspond to points in the COMPLEX PLANE, sometimes called an ARGAND PLANE after J. R. Argand, a French mathematician who first developed this relationship.

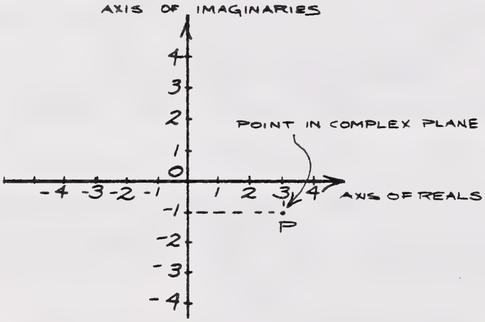
In the complex plane, the horizontal axis is called the REAL AXIS. All complex numbers of the form $a + 0i$ lie on the real axis. The vertical axis is called the IMAGINARY AXIS. All complex numbers of the form $0 + bi$ lie on this axis. Any point (x, y) in the plane represents the complex number $x + yi$. In other words, the abscissa of the point corresponds to the real part of the complex number and the ordinate corresponds to the coefficient of i.

There is a one-to-one correspondence between the set of complex numbers and the set of points in the complex plane. This means that for every point in the plane there is a corresponding complex number, and for every complex number, there is a corresponding point in the plane.

Suppose we wished to plot the complex number -2 + 3i in the complex plane. We could locate this point by finding the number -2 on the axis of reals and the number 3 on the axis of imaginaries. The point which lies directly above -2 on the axis of reals and directly to the left of 3 on the axis of imaginaries corresponds to the complex number -2 + 3i.



On the other hand, suppose we wished to find the complex number which corresponds to the point P in the complex plane below.



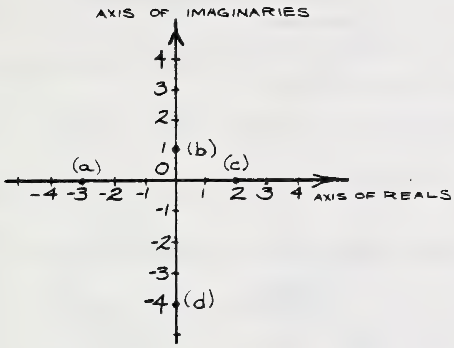
We can find the real part of the complex number by drawing an imaginary vertical line through P and noting that it cuts the axis of reals at 3. Thus, the real part of the complex number must be 3.

We can find the coefficient of i in the complex number by drawing an imaginary horizontal line through P and noting that it cuts the axis of imaginaries at -1. Thus, the coefficient of i in the complex number must be -1.

Thus, the complex number which corresponds to point P in the graph above is 3 - i.

All points on the horizontal axis correspond to real numbers and all points on the vertical axis correspond to pure imaginary numbers.

EXAMPLE:



Point (a) corresponds to the real number -3.

Point (b) corresponds to the pure imaginary number i.

Point (c) corresponds to the real number 2.

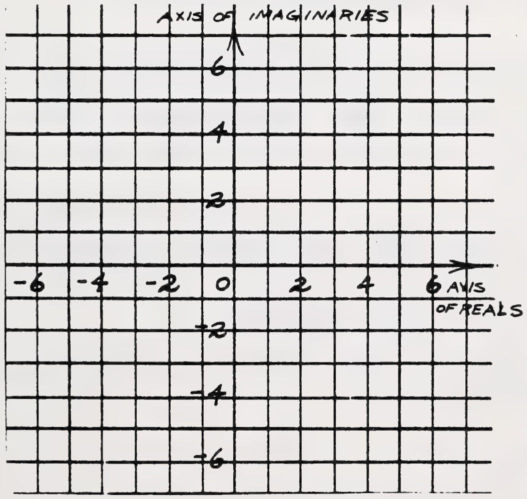
Point (d) corresponds to the pure imaginary number -4i.

Self-correcting Exercise #8

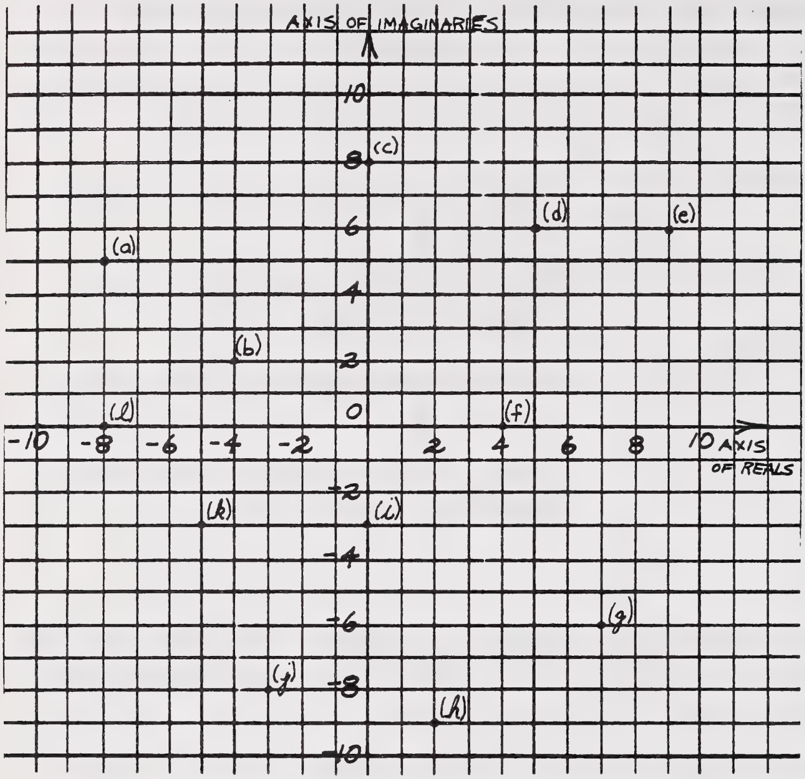
Answers may be found on page 62 of this lesson.

1. Graph the following complex numbers in the complex plane provided.

   -2i, 5, 4 + i, 2 - 3i, -2 - 4i, -2, -1 + 3i, 5i



2. State the complex number which corresponds to each lettered point.



(a) -8+5i

(b) __________

(c) __________

(d) __________

(e) __________

(f) __________

(g) __________

(h) __________

(i) __________

(j) __________

(k) __________

(l) __________

G. Properties of the Complex Number System

On page 49, lesson B, you found that the set of real numbers satisfies all the properties of a field under the operations of addition and multiplication. Addition and multiplication in set C have been defined in such a way that the field properties also apply in set C. Let us review these properties.

Fill in any blanks that appear on pages 25-27.

1. Set C is closed under addition.

This means that the sum of two complex numbers is always a complex number.

EXAMPLES:

$3i + 5i = 8i$ $\qquad 2 + (-5 + i) = -3 + i$ $\qquad (4 + 2i) + (-3 + 6i) = 1 + 8i$

Note that the results are all complex numbers.

2. Set C is closed under multiplication.

This means that the __________ of two complex numbers is always a

______________ number.

EXAMPLES:

2i × 3i = ___i^2 = ___

4(2 + i) = ________

(3 + i)(2 - i) = ________

3 × 5i = _____

Are all the results complex numbers? ________

3. Addition is commutative in set C.

This means that the order in which we __________ two complex numbers does not affect the result.

EXAMPLE:

(3 + 5i) + (2 + 4i) $\overset{?}{=}$	(2 + 4i) + (3 + 5i)
(3 + 2) + (5i + 4i)	(2 + 3) + (4i + 5i)
_____ + _____	_____ + _____

4. Multiplication is commutative in set C.

This means that the __________ in which we __________ two complex numbers does not affect the result.

EXAMPLE:

(3 + i)(2 - 5i) $\overset{?}{=}$	(2 - 5i)(3 + i)
6 + ___i - ___i - ___i^2	6 - ___i + ___i - ___i^2
6 - ___i + ___	6 - ___i + ___
___ - ___i	___ - ___i

5. Addition is associative in set C.

This means that the manner in which we group terms in a sum does not affect the result.

EXAMPLE:

3i + [(2 + 4i) + 5] $\overset{?}{=}$	[3i + (2 + 4i)] + 5
3i + [(2 + 5) + 4i]	[(3i + 4i) + ___] + 5
3i + (___ + 4i)	(___ + ___) + 5
___ + ___i	___ + ___i

6. Multiplication is associative in set C.

This means that the manner in which we group factors in a

____________ does not affect the result.

EXAMPLE: $i[(2 + i)(3 - i)] \overset{?}{=} [i(2 + i)](3 - i)$

$i(6 + \_\_i - \_\_i - i^2)$	$(\_\_i + i^2)(3 - i)$
$i(6 + \_\_ + 1)$	$(\_\_i - 1)(3 - i)$
$i(\_\_ + i)$	$6i - 3 - 2i^2 + i$
$\_\_ + i^2$	$7i - 3 + \_\_$
$\_\_ - 1$	____________

7. Multiplication distributes over addition in set C.

This means that a multiplier can be distributed over a pair of addends.

EXAMPLE: $-2i[(3 + i) + (4 - 3i)] \overset{?}{=} (-2i)(3 + i) + (-2i)(4 - 3i)$

Complete.

8. The additive identity in set C is the real number 0.

This means that when zero is added to any complex number, the number is left unchanged.

EXAMPLES:

$(3 + 2i) + 0 =$ _______ $-5 + 0 =$ _______ $15i + 0 =$ _______

In each case, is the number left unchanged? ____________

9. The multiplicative identity in set C is the real number 1.

This means that when any complex number is ____________ by one, the number is left ____________.

EXAMPLES:

$1 \times (3 + 2i) =$ ______ $1 \times 4i =$ ______ $1 \times (-2 + 5i) =$ ______

In each case, is the number left unchanged? ________

10. Every element in set C has an additive inverse.

For any complex number $a + bi$, its additive inverse is $-a - bi$. The sum of a complex number and its additive inverse is always zero.

EXAMPLES:

(i) $3 + 2i$ and $-3 - 2i$ are additive inverses since $(3 + 2i) + (-3 - 2i) = 0$.

(ii) $2 - 5i$ and _____ are additive inverses since (_____) $+ (-2 + 5i) = 0$.

(iii) $3i$ and ____ are additive inverses since $3i + (\quad) = 0$.

11. Every element in set C has a multiplicative inverse.

For any complex number $a + bi$, its multiplicative inverse is $\frac{1}{a + bi}$.

The product of a complex number and its ____________________ inverse is always one.

EXAMPLES:

(i) $2 - i$ and $\frac{1}{2 - i}$ are multiplicative inverses since $(2 - i) \times \frac{1}{(2 - i)} = 1$.

(ii) $1 + 5i$ and _____ are multiplicative inverses since $(1 + 5i) \times \frac{\quad}{1 + 5i} = 1$.

(iii) $2i$ and _____ are multiplicative inverses since $2i \times$ —— $= 1$.

Thus, like set Q and set R, set C satisfies all the eleven properties of a field. But, Q and R form ORDERED FIELDS while set C does not. You will recall that all real numbers are ordered. For any two real numbers a and b, either $a = b$, $a > b$, or $a < b$.

In set C, we can define equality for complex numbers, but we cannot define any order relation $>$ or $<$.

Two complex numbers are equal if their real parts are equal and their imaginary parts are equal.

DEFINITION - EQUALITY IN C

For two complex numbers $a + bi$ and $c + di$, $a + bi = c + di$ if and only if $a = c$ and $b = d$.

EXAMPLE: Find the values of x and y if $2x - 3yi = 4 + 9i$.

Solution

If $2x - 3yi = 4 + 9i$, then we know that:

$$2x = 4 \qquad \text{and} \qquad -3y = 9$$
$$x = 2 \qquad\qquad\qquad y = -3$$

In set C, a statement such as $4 + 5i > 3 + 6i$ is meaningless. All that can be said of the two complex numbers concerned is that $4 + 5i \neq 3 + 6i$. Similarly, one does not speak of a complex number such as $-3 + 4i$ as being positive or negative. These terms have no meaning for complex numbers, except for those of the form $a + 0i$ that are real numbers.

EXERCISE - The Complex Number System

1. Fill in the blanks.

(a) The set of ________________ numbers was invented in order to solve equations like $-3x = 7$.

(b) The ________________ inverse of $5 - 2i$ is $-5 + 2i$.

(c) The ________________ inverse of $3 + i$ is $\frac{1}{3 + i}$.

(d) All complex numbers of the form bi where $b \in R$ are called ______ __________ numbers. They correspond to points on the axis of ________________ in the complex plane.

(e) Any complex number can be written in the form __________, where a and b are ________________ numbers.

(f) The set of real numbers is a __________________ of the set of complex numbers.

(g) By definition, i = ______ and i^2 = _____.

(h) The two complex numbers $5x + yi$ and $10 - 5i$ are equal if and only if x = ______ and y = ______.

(i) The set of ____________________ numbers was invented to provide roots for equations like $x^2 - 5 = 0$.

(j) The product of a complex number and its ______________________ is always a real number.

(k) For any complex numbers $(x + yi)$ and $(a - bi)$,

$$(x + yi) - (a - bi) = (x + yi) + (\_\_\_\_\_\_\_\_).$$

(l) The roots of any polynomial equation can be found in the set of __________________ numbers.

2. Add the following pairs of complex numbers.

$4 + 3i$, $-7 + 2i$ $(4+3i) + (-7+2i)$ $= (4-7) + (3i+2i)$ $= -3+5i$	$6 - 3i$, $-2 - 5i$	5, $-9 + 4i$
$3i$, $-7 + 8i$	$3 + 6i$, $4 - i$	$-3 - i$, $5 + i$

3. In each case, subtract the second complex number from the first.

$2 + 3i, \quad -5 + 7i$ $(2+3i) - (-5+7i)$ $= (2+3i) + (5-7i)$ $= (2+5) + (3i-7i)$ $= 7 - 4i$	$6 + 4i, \quad 3 - i$	$-2i, \quad 5 + 4i$
$\sqrt{2} + 3i, \quad 5\sqrt{2} - i$	$2 - i, \quad 3$	$5 + 4i, \quad -3 - 8i$

4. Multiply.

(a) $3i \times -2i =$ __________

(b) $i \times 7i =$ __________

(c) $-3 \times 4i =$ __________

(d) $(4 + 3i)(7 + 2i) =$ $28 + 21i + 8i + 6i^2 = 28 + 29i - 6 = 22 + 29i$

(e) $(6 - 3i)(-2 + 4i) =$ __________

(f) $(2 - 5i)(2 + 5i) =$ __________

(g) $2i(-4 + 3i) =$ __________

(h) $-5(2 - 6i) =$ __________

(i) $(3 - i)(2 - 5i) =$ __________

(j) $(4\sqrt{2} - 3i)(\sqrt{2} + 2i) =$ __________

5. Divide. (Put answers to parts (b) to (e) in the form a + bi.)

(a) $\frac{-2}{i} =$

(b) $\frac{3}{2 - i} = \frac{3}{2 - i} \times \frac{2 + i}{2 + i} =$

(c) $\frac{6 - 2i}{1 + i} =$

(d) $\frac{2 + i}{4 - 3i} =$

(e) $\frac{3i}{4 + 2i} =$

6. Find the complex roots of the following equations. (Replace $\sqrt{-1}$ by i.)

(a) $x^2 + 5x + 8 = 0$

(b) $2x^2 + 8 = 0$

(c) $6x^2 - x + 2 = 0$

(d) $3x^2 + 2ix - 1 = 0$

7. Simplify the following expressions.

(a) $5(3 - 2i)^2$

=

(b) $\dfrac{i^3 + i^4}{1 + 2i^2}$

=

(c) $2i(1 + i) + 4(3 - 2i)$

=

(d) $(2 + i)(2 - i) - (5 - i)$

=

(e) $-5(1 - i) + 5 + 5i^3$

=

(f) $i^{24} \times (2i)^5$

=

(g) $(3 + 2i)(4 - i) - (5 + i)(7 - i)$

=

(h) $(3 + i)^4$

= $(3 + i)^2(3 + i)^2$

=

8. Evaluate $x^2 - 4x$ when $x = 2 + i$.

9. Show by substitution that $\frac{1 + i}{2}$ is a root of $2x^2 - 2x + 1 = 0$.

When $x = \frac{1 + i}{2}$, $2x^2 - 2x + 1$

$= 2(\qquad)^2 - 2(\qquad) + 1$

$=$

10. Decide if each of the following statements is true or false.

	True or False?
(a) $(-i)^2 = -1$	________
(b) $(3i)^2 = 9$	________
(c) $\sqrt{-2} \times \sqrt{-3} = \sqrt{6}$	________
(d) $i^{27} = i^{31}$	________
(e) $\sqrt{-7} = 7i$	________
(f) There is a one-to-one correspondence between set C and the points in a plane.	________
(g) Set C is closed under addition, subtraction, multiplication and division (except for division by zero).	________
(h) Sets Q, R, and C all form fields with respect to the operations of addition and multiplication.	________
(i) In set C, $a + bi > c + di$ if $a > c$ and $b > d$.	________

Topic Two: Solving Systems of Equations

A. Systems of Two Linear Equations in Two Variables

$$\begin{cases} x + y = 2 \\ x + 2y = 6 \end{cases}$$

is called a system of two linear equations in two variables. In order to find the solution set of such a system, we must look for ordered pairs of values of x and y which are common to the solution set of each equation in the system. Any ordered pair which belongs to the solution set of $x + y = 2$ and also belongs to the solution set of $x + 2y = 6$, will belong to the solution set of the system.

We will now discuss three different ways of solving linear systems.

I. Graphical Method

The solution set of a system of two linear equations in two variables can be determined graphically by drawing the straight-line graphs of the two equations on the same set of axes and then noting the coordinates of the point where the two graphs intersect. This ordered pair will belong to the solution set of the system since it is common to the solution sets of the two equations.

EXAMPLE: $$\begin{cases} x + y = 2 & (1) \\ x + 2y = 6 & (2) \end{cases}$$

Solution

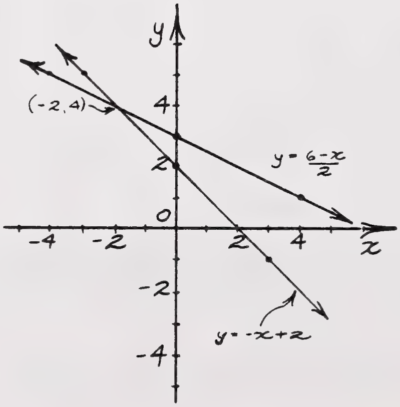
Isolate y on the left side of each equation, set up tables of values, and then draw the two straight-line graphs.

$y = -x + 2$ (1)

x	-3	0	3
y	5	2	-1

$$y = \frac{6 - x}{2}$$

x	-4	0	4
y	5	3	1



Note that the two straight-line graphs intersect at the point whose coordinates are (-2, 4). Therefore, the solution set of the system is $\{(-2, 4)\}$.

Solve the following system of equations graphically.

$$\begin{cases} 2x - y = 3 & (1) \\ x + y = 6 & (2) \end{cases}$$

Begin by isolating y on the left side of each equation and then set up tables of values.

$2x - y = 3$ (1)

$-y = 3 -$ _____

Multiply every term by -1.

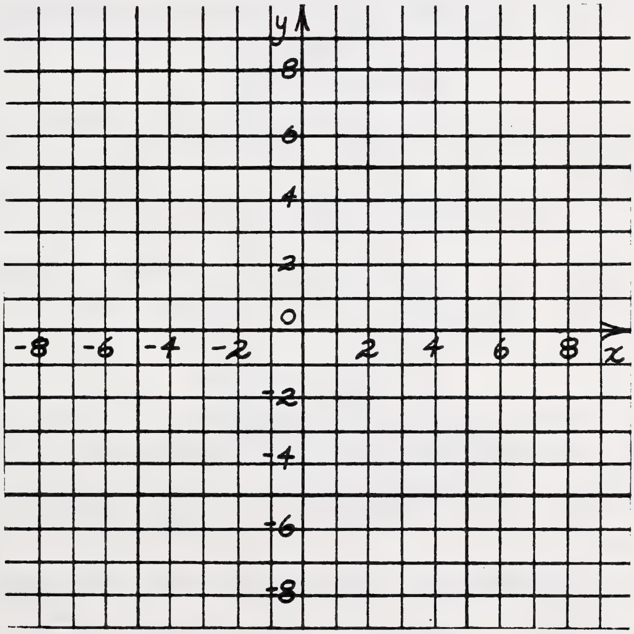
$\boxed{y = \_\_\_\_\_\_}$

x	-2	0	2	4
y				

$x + y = 6$ (2)

$\boxed{y = \_\_\_\_\_\_}$

x	-2	0	4	6
y				



The two straight lines intersect at the point whose coordinates are (_____,_____). Thus, the solution set of the system is {(_____,_____)}. This means that both equations are satisfied when x = _____ and y = _____.

II. Elimination Method

The solution set of a system of linear equations in two variables can be determined algebraically by finding an equivalent system whose solution set can be determined by inspection. Two systems of equations are equivalent if they have the same solution set.

The elimination method of solving a system of equations makes use of the fact that an equivalent system is obtained if either equation in the original system is replaced by an equation which represents the sum or difference of the two equations in the system.

EXAMPLE:

$$\begin{cases} x + y = 2 & (1) \\ x + 2y = 6 & (2) \end{cases}$$

Solution

Apply the subtraction property of equality and subtract equation (2) from equation (1).

$$\begin{array}{rcrl} x + y &=& 2 & (1) \\ \underline{x + 2y} &\underline{=}& \underline{6} & \underline{(2)} \\ -y &=& -4 & \quad \text{(subtract)} \\ \therefore y &=& 4 & (3) \end{array}$$

Now keep equation (1) in the system and replace equation (2) by equation (3). The new system which contains equations (1) and (3) is equivalent to the original system.

$$\begin{cases} x + y = 2 & (1) \\ \phantom{x +{}} y = 4 & (3) \end{cases}$$

In order to solve this new system, substitute 4 for y in equation (1) and solve for x.

$$\begin{array}{rcrl} x + 4 &=& 2 & \\ \therefore x &=& -2 & (4) \end{array}$$

Now, keep equation (3) in the system and replace equation (1) by equation (4). This new system which contains equations (3) and (4) is equivalent to the original system.

$$\begin{cases} x = -2 & (4) \\ y = 4 & (3) \end{cases}$$

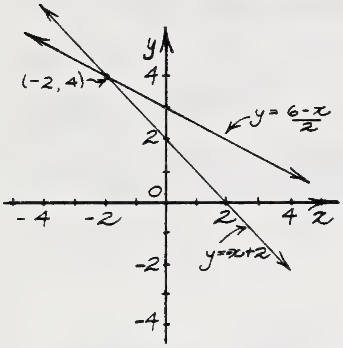
By inspection, we can see that the solution set of this system is $\{(-2, 4)\}$. Since this derived system is equivalent to the original system, the solution set of the original system must also be $\{(-2, 4)\}$. Note that on page 34 of this lesson we obtained the same solution set graphically.

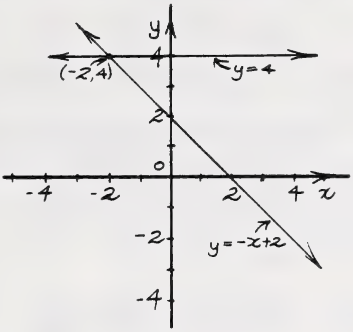
Although it is not immediately obvious that the three systems

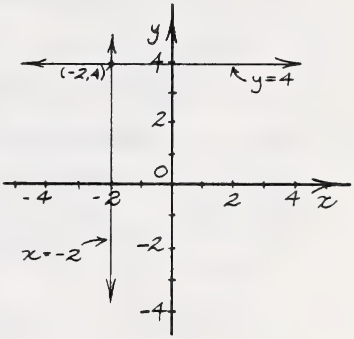
$$\begin{cases} x + y = 2 & (1) \\ x + 2y = 6 & (2) \end{cases} \qquad \begin{cases} x + y = 2 & (1) \\ \phantom{x +{}} y = 4 & (3) \end{cases} \qquad \begin{cases} x = -2 & (4) \\ y = 4 & (3) \end{cases}$$

are equivalent, it can be demonstrated graphically that all three systems have the solution set $\{(-2, 4)\}$.

$$\begin{cases} x + y = 2 & (1) \\ x + 2y = 6 & (2) \end{cases} \qquad \begin{cases} x + y = 2 & (1) \\ y = 4 & (3) \end{cases} \qquad \begin{cases} x = -2 & (4) \\ y = 4 & (3) \end{cases}$$







When solving a system of equations by the elimination method, you should set down your work in the following manner:

$$\begin{array}{rcll} x + y &=& 2 & (1) \\ \underline{x + 2y} &\underline{=}& \underline{6} & (2) \\ -y &=& -4 & \text{(subtract)} \\ \therefore y &=& 4 & \end{array}$$

Substitute 4 for y in (1).

$$\begin{array}{rcl} x + 4 &=& 2 \\ \therefore x &=& -2 \end{array}$$

The solution set is $\{(-2, 4)\}$.

You will sometimes find that you will have to add the two equations in order to obtain an equation in only one variable.

EXAMPLE:

$$\begin{cases} 2x + y = 5 & (1) \\ x - y = 7 & (2) \end{cases}$$

Solution

$$\begin{array}{rcll} 2x + y &=& 5 & (1) \\ \underline{x - y} &\underline{=}& \underline{7} & (2) \\ 3x &=& 12 & \text{(add)} \\ \therefore x &=& 4 & \end{array}$$

Substitute 4 for x in (2).

$$\begin{array}{rcl} 4 - y &=& 7 \\ -y &=& 3 \\ \therefore y &=& -3 \end{array}$$

The solution set is $\{(4, -3)\}$.

In general, subtract the two equations whenever either the x-terms or the y-terms in the two equations are identical. For example, in each of the systems of equations below, you must subtract the two equations in order to obtain an equation in one variable.

$$\begin{cases} 3x + 2y = 7 \\ 3x + 5y = 8 \end{cases}$$

(Identical terms)

$$\begin{cases} x - 4y = 1 \\ 2x - 4y = 3 \end{cases}$$

(Identical terms)

$$\begin{cases} x + y = 3 \\ 2x + y = 5 \end{cases}$$

(Identical terms)

Add the two equations whenever either the x-terms or the y-terms in the two equations are additive inverses of each other. For example, in each of the systems of equations below, you would add the two equations in order to obtain an equation in one variable.

$$\begin{cases} 2x - 5y = 3 \\ x + 5y = 7 \end{cases}$$

(Additive inverses)

$$\begin{cases} 3x + 2y = 9 \\ -3x + 4y = 1 \end{cases}$$

(Additive inverses)

$$\begin{cases} x + y = 3 \\ 3x - y = 5 \end{cases}$$

(Additive inverses)

Self-correcting Exercise #9

Answers may be found on page 62 of this lesson.

1. Tell whether you would add or subtract the two equations in order to obtain an equation in one variable.

(a) $\begin{cases} 8x + 3y = 7 & (1) \\ 5x - 3y = 4 & (2) \end{cases}$

Add or subtract? ____________

(b) $\begin{cases} 2x - 3y = 6 & (1) \\ -2x + 7y = 2 & (2) \end{cases}$

Add or subtract? ____________

(c) $\begin{cases} 4m - 3n = 2 & (1) \\ 4m + 7n = 8 & (2) \end{cases}$

Add or subtract? ____________

(d) $\begin{cases} 6a + b = 5 & (1) \\ 3a + b = 1 & (2) \end{cases}$

Add or subtract? ____________

2. Find the solution set of each system by the elimination method.

(a) $\begin{cases} 11x - y = 42 & (1) \\ 3x + y = 0 & (2) \end{cases}$

(b) $\begin{cases} 4x + 2y = 1 & (1) \\ 4x + 3y = -5 & (2) \end{cases}$

In some systems of linear equations, neither the x-terms nor the y-terms are identical or additive inverses of each other. In such cases, you must use the multiplication property of equality to make a pair of corresponding terms either identical or additive inverses.

EXAMPLE:

$$\begin{cases} 2x + 5y = 28 & (1) \\ 3x + 2y = 31 & (2) \end{cases}$$

Solution. Note that if we were to multiply every term in equation (1) by 3 and every term in equation (2) by 2, we would obtain an equivalent system which has identical x-terms.

$$\begin{array}{rcll} 6x + 15y &=& 84 & (1) \times 3 \\ \underline{6x + \;\;4y} &=& \underline{62} & (2) \times 2 \\ 11y &=& 22 & \text{(subtract)} \\ \therefore y &=& 2 & \end{array}$$

Substitute 2 for y in (2).

$$\begin{array}{rcl} 3x + 2(2) &=& 31 \\ 3x + 4 &=& 31 \\ 3x &=& 27 \\ \therefore x &=& 9 \end{array}$$

The solution set is $\{(9, 2)\}$.

Self-correcting Exercise #10

Answers may be found on page 63 of this lesson.

1. Fill in the blanks.

(a) Consider the system $\begin{cases} 2x + 3y = -14 & (1) \\ x - y = 3 & (2) \end{cases}$

We could eliminate the variable x if we multiplied equation ___ by _____ and then ____________________ the two equations.

(b) Consider the system $\begin{cases} 3x - 2y = 10 & (1) \\ 5x + 3y = -15 & (2) \end{cases}$

We could eliminate the y's if we multiplied equation (1) by _____, multiplied equation (2) by _____ and ____________________ the two equations.

(c) Consider the system $\begin{cases} -4x + 5y = 2 & (1) \\ 9x + 4y = 1 & (2) \end{cases}$

We could eliminate the y's if we multiplied equation (1) by _____, multiplied equation (2) by _____ and ____________________ the two equations.

2. Solve each system by the elimination method.

(a) $\begin{cases} 2x + 5y = -1 & (1) \\ 3x + 7y = 2 & (2) \end{cases}$

(b) $\begin{cases} a + 3b = 10 & (1) \\ 5a + 7b = 2 & (2) \end{cases}$

III. Substitution Method

The solution set of a system of linear equations in two variables can also be determined algebraically by the substitution method. Like the elimination method, the substitution method involves replacing the given system by a series of equivalent systems until the solution set can be determined by inspection. In the substitution method, we must replace one of the equations in the system by an equivalent equation in which either x or y is isolated on the left-hand side.

EXAMPLE: $\begin{cases} x + 4y = 3 & (1) \\ 2x - 3y = 17 & (2) \end{cases}$

In equation (1), use the subtraction principle of equality to isolate x on the left-hand side.

$$x + 4y = 3$$
$$\therefore x = 3 - 4y \quad (3)$$

Now, keep equation (2) in the system and replace equation (1) by equation (3). The new system which contains equations (2) and (3) is equivalent to the original system.

$$\begin{cases} x = 3 - 4y & (3) \\ 2x - 3y = 17 & (2) \end{cases}$$

In order to solve this new system, substitute 3 - 4y for x in equation (2) and solve for y.

$$\begin{aligned} 2(3 - 4y) - 3y &= 17 \\ 6 - 8y - 3y &= 17 \\ 6 - 11y &= 17 \\ -11y &= 11 \\ y &= -1 \quad (4) \end{aligned}$$

Now, keep equation (3) in the system and replace equation (2) by equation (4). This new system which contains equations (3) and (4) is equivalent to the original system.

$$\begin{cases} x = 3 - 4y & (3) \\ y = -1 & (4) \end{cases}$$

In order to solve this new system, substitute -1 for y in equation (3) and solve for x.

$$\begin{aligned} x &= 3 - 4(-1) \\ x &= 3 + 4 \\ x &= 7 \qquad (5) \end{aligned}$$

Keep equation (4) in the system and replace equation (3) by equation (5). This new system which contains equations (4) and (5) is equivalent to the original system.

$$\begin{cases} x = 7 & (5) \\ y = -1 & (4) \end{cases}$$

By inspection, we can see that the solution set of this system is $\{(7, -1)\}$. Since this derived system is equivalent to the original system, the solution set of the original system must also be $\{(7, -1)\}$.

In solving a system of equations by the substitution method, you should set down your work in the following manner:

$$\begin{cases} x + 4y = 3 & (1) \\ 2x - 3y = 17 & (2) \end{cases}$$

From equation (1), $x + 4y = 3$

$$\therefore x = 3 - 4y \qquad (3)$$

Substitute 3 - 4y for x in equation (2).

$$\begin{aligned} 2(3 - 4y) - 3y &= 17 \\ 6 - 8y - 3y &= 17 \\ 6 - 11y &= 17 \\ -11y &= 11 \\ y &= -1 \end{aligned}$$

Substitute -1 for y in equation (3).

$$\begin{aligned} x &= 3 - 4(-1) \\ x &= 3 + 4 \\ x &= 7 \end{aligned}$$

The solution set is $\{(7, -1)\}$.

When using the substitution method to solve a system of equations, first look at the variable terms in the equations to see if there is one with a numerical coefficient of 1 or -1. Your work will be simplified if you isolate the variable in that term on one side of the equation. For example, in the system above, equation (1) contains the term x. Therefore, we isolated x on the left side and obtained the equation $x = 3 - 4y$.

Self-correcting Exercise #11

Answers may be found on page 64 of this lesson.

1. Find the solution set of each system by the substitution method.

(a) $\begin{cases} 3x + 2y = 29 \\ y - x = -8 \end{cases}$ (b) $\begin{cases} 3x - 4y = 0 \\ 2x + y = 33 \end{cases}$

B. Independent, Inconsistent and Dependent Linear Systems

Linear systems in two variables may be classified into three types. These three types are discussed in the sections that follow.

I. Independent Systems

A system of two linear equations in two variables is called an INDEPENDENT SYSTEM if its solution set contains exactly one ordered pair. If we graph such a system, the two straigh-line graphs will intersect at exactly one point.

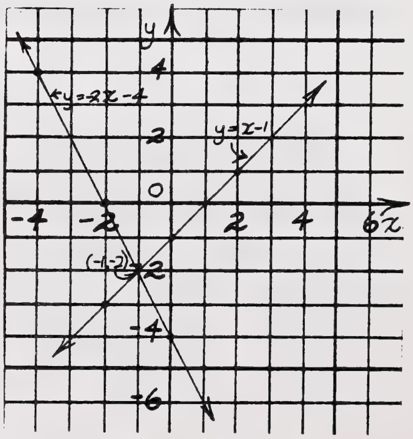
EXAMPLE: $\begin{cases} x - y - 1 = 0 & (1) \\ 2x + y + 4 = 0 & (2) \end{cases}$

$y = x - 1 \quad (1)$

x	-2	0	2
y	-3	-1	1

$y = -2x - 4 \quad (2)$

x	-4	-2	0
y	4	0	-4



The two straight-line graphs intersect at the one point (-1, -2). The solution set of the system is {(-1, -2)}.

This system could be solved algebraically using either the elimination or the substitution method.

Complete the solutions below.

Elimination Method

$x - y = 1$ (1)
$\underline{2x + y = -4}$ (2)

(add)

The solution set is {(___, ___)}.

Substitution Method

$x - y = 1$ (1)
$2x + y = -4$ (2)
From equation (1), x = __________.

Substitute this value for x in (2).

The solution set is __________.

II. Inconsistent Systems

A system of two linear equations in two variables is called an INCONSISTENT SYSTEM if its solution set is ϕ, the empty set. If we graph such a system, the two straight lines will be parallel (i.e. they will never intersect). This situation occurs when the x-terms in the two equations have identical coefficients, the y-terms have identical coefficients, and the constant terms are different.

EXAMPLE:

$x - y = 1$ (1)
$x - y = -2$ (2)

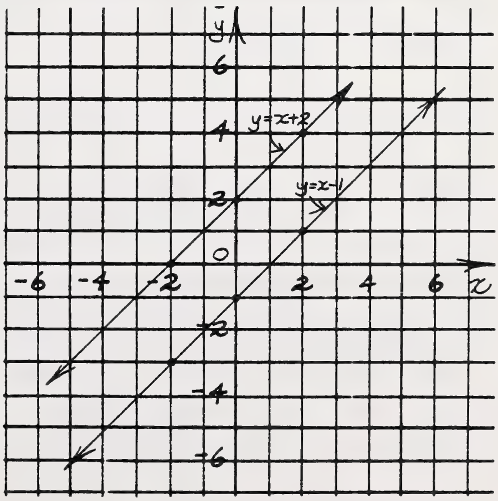
Complete the tables of values below.

$y = x - 1$ (1)

x	-2	0	2
y			

$y = x + 2$ (2)

x	-2	0	2
y			



The two straight-line graphs are parallel. The solution set of the system is ϕ. (In the equations, note that only the constant terms are different.)

When we try to solve this system algebraically, we end up with a numerical equation that is false. This signifies that no pair of values of x and y satisfies both equations.

Complete the solutions below.

Elimination Method

$$\begin{array}{rcrl} x - y &=& 1 & (1)\\ \underline{x - y} &\underline{=}& \underline{-2} & (2)\\ 0 &=& \underline{\quad} & \text{(subtract)}\end{array}$$

This statement is always false.
The solution set is ϕ.

Substitution Method

$$\begin{array}{rcrl} x - y &=& 1 & (1)\\ \underline{x - y} &\underline{=}& \underline{-2} & (2)\end{array}$$

From equation (1), x = __________.

Substitute this value of x in (2).

$$(\qquad) - y = -2$$

$$\underline{\qquad} = -2$$

This statement is always __________.

The solution set is __________.

III. Dependent Systems

A system of two linear equations in two variables is called a DEPENDENT SYSTEM if its solution set is an infinite number of ordered pairs. If we graph such a system, the two straight-line graphs will coincide. This situation occurs when the two equations are identical or one can derived from the other by multiplying through by the appropriate real number.

EXAMPLE:

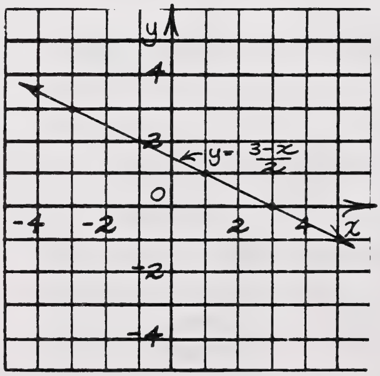
$$\begin{array}{rcl} x + 2y = 3 & (1)\\ 2x + 4y = 6 & (2)\end{array}$$

Both equations can be put in the form $y = \dfrac{3 - x}{2}$.

$$y = \frac{3 - x}{2}$$

x	-3	1	3
y			

Complete.



The two straight-line graphs coincide. The solution set is an infinite set of ordered pairs. For example, (-3, 3), (1, 1) and (3, 0) are three such ordered pairs. Note that equation (2) can be derived from equation (1) by multiplying every term in equation (1) by 2.

When we try to solve this system algebraically, we end up with a numerical equation that is true.

Complete the solutions below.

Elimination Method

$$\begin{aligned} x + 2y &= 3 \quad (1) \\ 2x + 4y &= 6 \quad (2) \end{aligned}$$

$$\begin{aligned} 2x + 4y &= 6 \quad (1) \times 2 \\ \underline{2x + 4y} &\underline{= 6} \quad (2) \\ 0 &= \underline{\quad} \quad \text{(subtract)} \end{aligned}$$

This statement is always true. The solution set contains an infinite number of ordered pairs.

Substitution Method

$$\begin{aligned} x + 2y &= 3 \quad (1) \\ 2x + 4y &= 6 \quad (2) \end{aligned}$$

From equation (1), $x = \underline{\qquad\qquad}$.

Substitute this value for x in (2).

$$\begin{aligned} 2(\qquad\qquad) + 4y &= 6 \\ 6 - \underline{\qquad} + 4y &= 6 \\ \underline{\qquad} &= 6 \end{aligned}$$

This statement is always $\underline{\qquad\qquad}$.

The solution set is $\underline{\qquad\qquad\qquad}$

$\underline{\qquad\qquad\qquad\qquad\qquad}$.

Self-correcting Exercise #12

Answers may be found on page 64 of this lesson.

1. By inspection, tell whether each system has one real solution, no real solution, or an infinite number of solutions. Then classify the system as being independent, inconsistent, or dependent.

System	Number of Solutions	Classification
(a) $\begin{cases} x - y = 0 \\ 2x = 2y \end{cases}$		
(b) $\begin{cases} 3y = x - 4 \\ 2x - 1 = y + 7 \end{cases}$		
(c) $\begin{cases} y = 3x + 1 \\ 2y = 6x + 2 \end{cases}$		
(d) $\begin{cases} x + 4y = 7 \\ x + 4y = -2 \end{cases}$		
(e) $\begin{cases} x = y \\ x = 2y \end{cases}$		
(f) $\begin{cases} x = 2y + 3 \\ x - 2y = 4 \end{cases}$		

C. Linear-Quadratic Systems of Equations

A set of two equations, one of the first degree and the other of the second degree, each in two unknowns, is called a LINEAR-QUADRATIC SYSTEM. Linear-quadratic systems can be solved both graphically and algebraically.

I. Graphical Solutions

In previous lessons, you learned that the graph of a linear equation in two variables is a straight line, while the graph of a quadratic equation in two variables is a curve (a parabola, circle, ellipse or hyperbola).

When a linear-quadratic system is graphed, we find that the straight line and curve may be related in three different ways. They may intersect in two points, one point, or no points. These three cases are illustrated in the examples below.

Fill in any blanks that appear in the examples.

EXAMPLE 1: $x + y = 7 \quad (1)$

$xy = 12 \quad (2)$

Solution

The graph of equation (1) is

a ______________________.

$y = 7 - x \quad (1)$

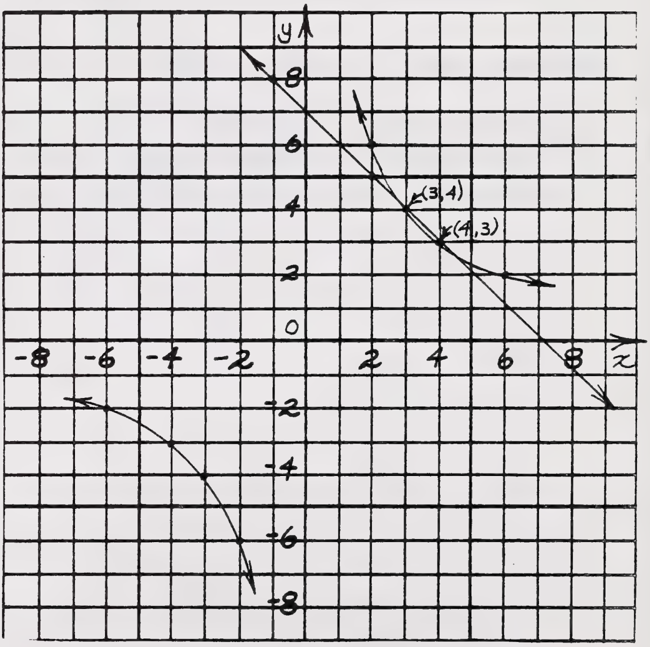
x	-1	2	4
y			

The graph of equation (2) is a

______________________.

$y = \dfrac{12}{x} \quad (2)$

x	-6	-4	-3	-2	2	3	4	6
y	-2							2



The straight-line graph intersects the curve at the two points (3, 4) and

(___, ___). The solution set of the system is {(3, 4), (___, ___)}.

EXAMPLE 2:

$$4x + 3y - 25 = 0 \quad (1)$$
$$x^2 + y^2 = 25 \quad (2)$$

Solution

The graph of equation (1) is

a ______________________________.

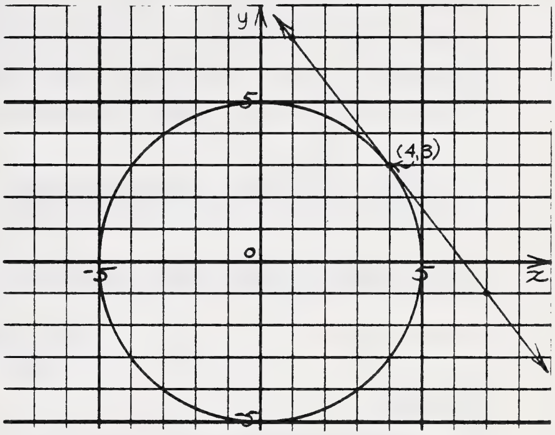
$$y = \frac{\qquad}{3} \quad (1)$$

x	1	4	7
y			

The graph of equation (2) is

a ______________________ with

a radius of ______.



The straight-line graph is tangent to the curve. (This means that it "touches" the curve in exactly one place.) The point of tangency is (4, 3). The solution set of the system is {(4, 3)}.

EXAMPLE 3:

$$2x + y + 8 = 0 \quad (1)$$
$$y - x^2 + x = -6 \quad (2)$$

Solution

The graph of equation (1)

is a _______________________.

$$y = -2x - \underline{\qquad} \quad (1)$$

x	-5	-3	-1
y			

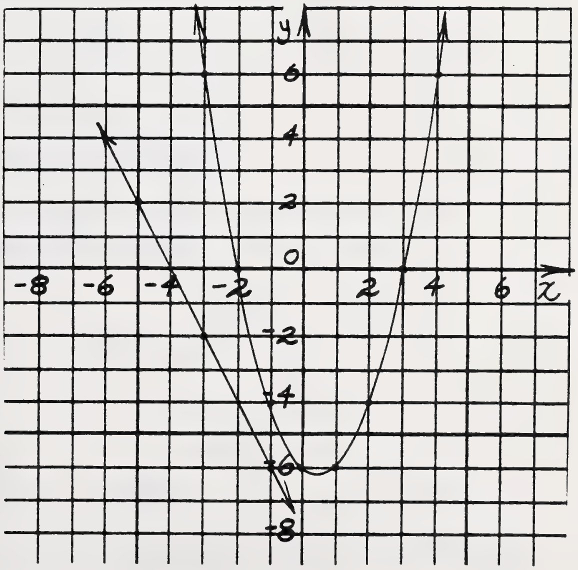
The graph of equation (2)

is a _______________________

which opens _______________.

$$y = x^2 - x - 6 \quad (2)$$

x	-3	-2	-1	0	1	2	3	4
y								



The straight-line graph never comes in contact with the curve. No ordered pairs from R × R satisfy both equations. The solution set of the system is ϕ, the empty set. (Roots are imaginary.)

II. Algebraic Solutions

A linear-quadratic system may be solved algebraically by using the substitution method. If we substitute from the linear equation into the quadratic, we can reduce the system to a single quadratic equation in one variable. You will recall that the roots of such an equation are either real and distinct, real and equal, or imaginary.

We will now look at the equation systems in Examples 1-3 again and this time solve them algebraically.

Fill in any blanks that appear in the examples.

EXAMPLE 1:

$$x + y = 7 \quad (1)$$
$$xy = 12 \quad (2)$$

Solution

From equation (1), $x + y = 7$

$$y = 7 - x \quad (3)$$

We can substitute $(7 - x)$ for y in equation (2).

SUBSTITUTE FROM THE LINEAR INTO THE QUADRATIC.

$$x(7 - x) = 12$$
$$7x - \_\_\_ = 12$$
$$x^2 - 7x + \_\_\_ = 0 \leftarrow \text{QUADRATIC EQUATION IN ONE UNKNOWN}$$

This equation can be solved by factoring the left member.

$$(x - 4)(\qquad) = 0$$
$$\underline{x = 4} \text{ or } \underline{x = 3}$$

We can substitute these values of x into equation (3) in order to find the corresponding values of y.

When $x = ④$

$y = 7 - x$

$y = 7 - 4$

$y = ③$

When $x = ③$

$y = 7 - x$

$y = 7 - 3$

$y = ④$

The solution set is $\{(4, 3), (3, 4)\}$.

CAUTION: When writing the solution set, make sure that you have paired the values of x and y properly.

Note that this solution set agrees with the one obtained graphically in example 1 on page 46.

EXAMPLE 2:

$$\begin{aligned} 4x + 3y - 25 &= 0 \quad (1) \\ x^2 + y^2 &= 25 \quad (2) \end{aligned}$$

Solution

From equation (1),

$$\begin{aligned} 4x + 3y - 25 &= 0 \\ 3y &= -4x + \underline{\qquad} \\ y &= \frac{\qquad\qquad}{3} \quad (3) \end{aligned}$$

We can substitute $\frac{\qquad\quad}{3}$ for y in equation (2).

$$x^2 + \left(\frac{-4x + 25}{3}\right)^2 = 25$$

$$x^2 + \left(\frac{16x^2 - 200x + 625}{9}\right) = 25$$

Multiply every term by 9.

$$\begin{aligned} \underline{\qquad}x^2 + 16x^2 - 200x + 625 &= 225 \\ 25x^2 - \underline{\qquad}x + \underline{\qquad} &= 0 \end{aligned}$$

Divide every term by 25.

$$x^2 - 8x + 16 = 0 \longleftarrow \text{QUADRATIC EQUATION IN ONE UNKNOWN}$$

This equation can be solved by factoring the left member.

$$(x - 4)(\qquad\quad) = 0$$

$$x = 4$$

We can substitute this value of x into equation (3) in order to find the corresponding value of y.

When $x = 4$,

$$\begin{aligned} y &= \frac{-4x + 25}{3} \\ &= \frac{-4(4) + 25}{3} \\ &= \underline{\qquad\quad} \end{aligned}$$

The solution set is $\{(4, 3)\}$.

Note that this solution set agrees with the one obtained graphically in example 2 on page 47.

EXAMPLE 3: $2x + y + 8 = 0$ (1)
$y - x^2 + x = -6$ (2)

Solution

From equation (1), $2x + y + 8 = 0$

$y =$ ____________

We can substitute ______________ for y in equation (2).

$$(-2x - 8) - x^2 + x = -6$$
$$-x^2 - x - 2 = 0$$

Multiply every term by -1.

__________________ $= 0$

Use the quadratic formula to find the roots of this equation.

In this case, a = ______, b = ______, and c = ______.

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm\sqrt{(1)^2 - 4(1)(2)}}{2(1)} = \underline{\hspace{6cm}}$$

The values of x are not real.
If $x, y \in R$, the solution set is ϕ. (The roots are imaginary.)

Note that this solution set agrees with the one obtained graphically in example 3 on page 47.

Self-correcting Exercise #13

Answers may be found on page 64 of this lesson.

1. Classify the graphs that correspond to the equations in each system.

(a) $2x + 4y = 3$ (1)
$3x^2 - y^2 = 5$ (2)

The graph of (1) is a

____

The graph of (2) is a

____

(b) $y = 2x$ (1)
$x - 5y = 7$ (2)

The graph of (1) is a

____

The graph of (2) is a

____

(c) $y = x^2 - 5$ (1)
$y = x + 3$ (2)

The graph of (1) is a

____

The graph of (2) is a

____

(d) $2x^2 + y^2 = 49$ (1)
$x - 5y = 3$ (2)

The graph of (1) is a

____

The graph of (2) is a

____

EXERCISE - Solving Systems of Equations

1. For each of the following linear systems, graph each equation and tell whether the system is independent, inconsistent or dependent. Then state the solution set.

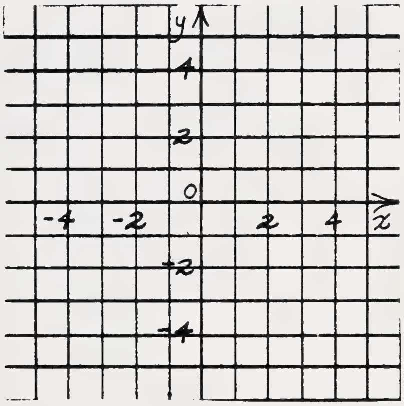
(a) $x + y = 0$ (1)
$2x + 3y = 3$ (2)

$y =$ ____________ (1)

x	-2	0	2
y	2		

$y = \frac{\_\_\_\_\_\_}{3}$ (2)

x	-3	0	3
y			

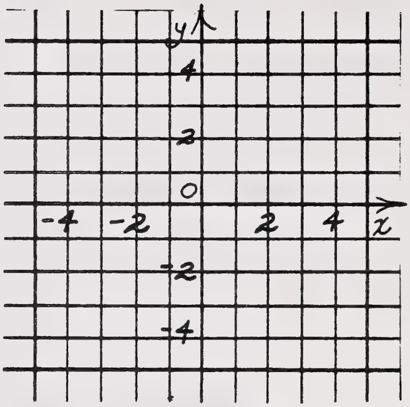


The system is

_______________________.

The solution set is __________.

(b) $2y = x + 1$ (1)
$2x + 2 = 4y$ (2)



$y =$ __________ (1)

x	-3	-1	1
y			

$y =$ __________ (2)

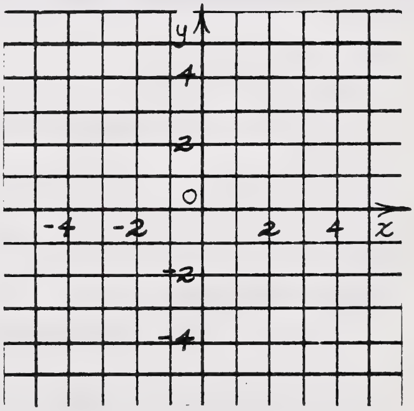
x	
y	

The system is

__________________.

The solution set is __________.

(c) $x - 2y = 0$ (1)
$2y - x = 2$ (2)

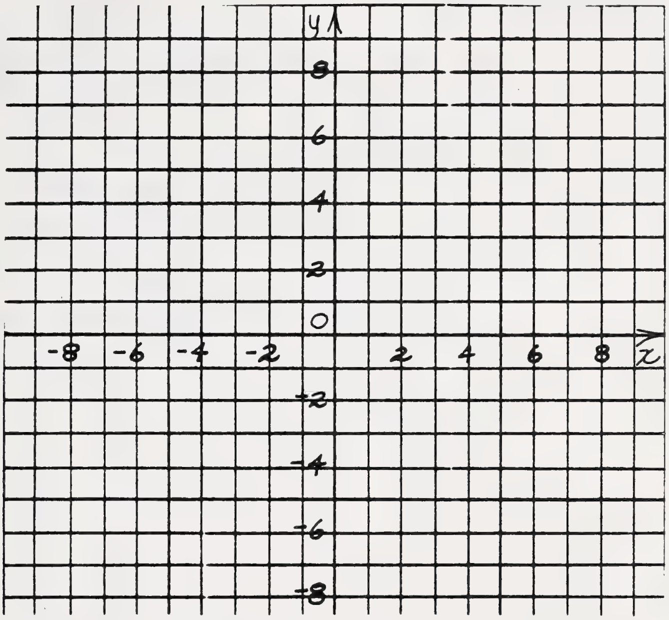


The system is

__________________.

The solution set is __________.

(d) $3y = x - 4$ (1)
$2x - 1 = y + 7$ (2)



The system is

____________________.

The solution set is ____________.

2. Solve each linear system by the elimination method.

(a) $3x + 4y = 2$ (1)
$2x - y = -6$ (2)

(b) $2x - 3y = -24$ (1)
$3x - 5y = 22$ (2)

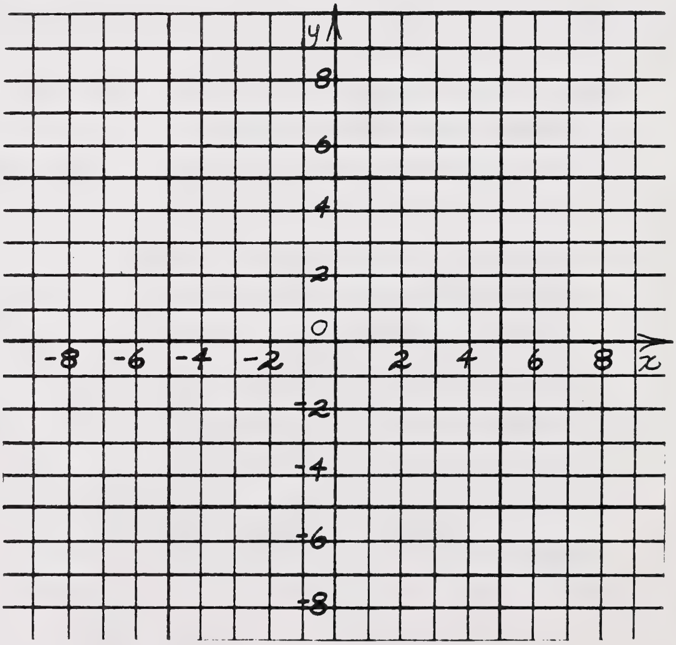
3. Solve each linear system by the substitution method.

(a) $3x + 6y = 13$ (1)
$9x + y = 5$ (2)

(b) $2x - y = 7$ (1)
$3x + 2y = 14$ (2)

4. Solve each linear-quadratic system both graphically and algebraically. (In each case, x and y are real numbers.)

(a) $x^2 - 2y^2 = 7$ (1)
$2y - x = 1$ (2)



Graphical Solution

The graph of equation

(1) is a ______________.

$$y = \pm\sqrt{\frac{x^2 - 7}{2}}$$

x		±3	±5	±7
y	0			

The graph of equation

(2) is a ______________.

$$y = \frac{\qquad}{2}$$

x	-7	-1	7
y			

How many points do the 2 graphs have in common? __________.

The solution set is __________.

$$x^2 - 2y^2 = 7 \quad (1)$$

$$2y - x = 1 \quad (2)$$

Algebraic Solution

From equation (2), $2y - x = 1$

$-x = -2y + 1$

$x =$ __________

Substitute ______ for x in equation (1).

$(\qquad)^2 - 2y^2 = 7$

Solve for y.

When y = _____, x =

When y = _____, x =

The solution set is ___________________.

(b) $x^2 + y^2 = 16$ (1)

$2y = x - 10$ (2)

Graphical Solution

The graph of equation (1)

is a __________ with a

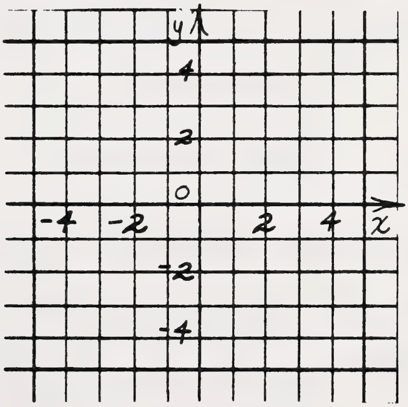
radius of ______.

The graph of equation (2)

is a ___________________.

$y = \frac{\qquad}{2}$

x	-2	2	4
y			



How many points do the 2 graphs have in common? ______.

The solution set is __________.

$$x^2 + y^2 = 16 \qquad (1)$$
$$2y = x - 10 \qquad (2)$$

Algebraic Solution

From equation (2), $x =$ _____ $+ 10$

Substitute __________ for x in equation (1).

$$(\qquad)^2 + y^2 = 16$$

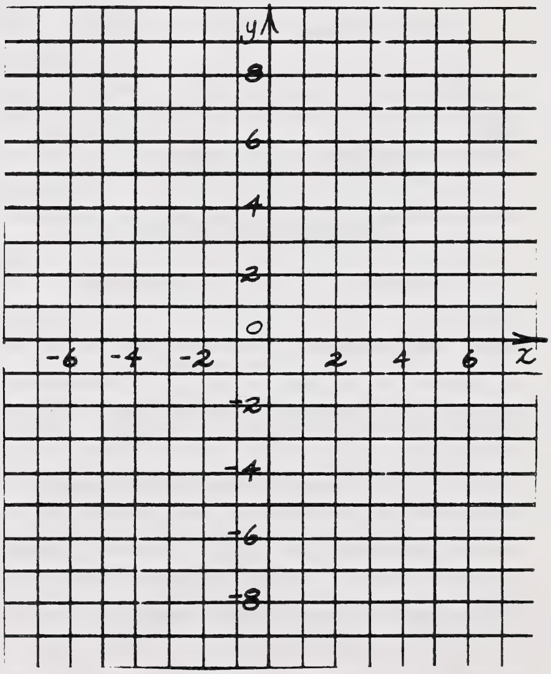
Solve for y by using the quadratic formula.

Are the roots real? ________

The solution set is ________ if $x, y \in R$.

(c) $2x + y + 4 = 0$ (1)

$4y = x^2$ (2)



Graphical Solution

The graph of equation (1) is

a ______________________________.

$y =$ __________

x	-6	-2	1
y			

The graph of equation (2) is

a ________________________ which

opens ________________________.

$y = \frac{\qquad}{4}$

x	0	±1	±2	±3	±4	±5	±6
y				2.25			

How many points do the two graphs have in common? __________.

The solution set is __________.

$$\begin{aligned} 2x + y + 4 &= 0 \quad (1) \\ 4y &= x^2 \quad (2) \end{aligned}$$

Algebraic Solution

From equation (1), $y =$ __________.

Substitute __________ for y in equation (2).

$$4(\qquad\qquad) = x^2$$

Solve for x. $\Bigg\{$

When $x =$ _____, $y =$

The solution set is __________.

Key to Self-correcting Exercises in Lesson I

Exercise #1, page 3

1.

(a) $\sqrt{-2} = \sqrt{2 \times -1} = \sqrt{2} \times \sqrt{-1} = \boxed{\sqrt{2}i}$

(b) $3 + \sqrt{-5} = \boxed{3 + \sqrt{5}i}$

(c) $\sqrt{-7} + \sqrt{-3} = \boxed{\sqrt{7}i + \sqrt{3}i}$

(d) $\frac{\sqrt{-9}}{7} = \frac{\sqrt{9 \times -1}}{7} = \frac{\sqrt{9}i}{7} = \boxed{\frac{3i}{7}}$

(e) $\frac{-2 \pm \sqrt{-4}}{5} = \boxed{\frac{-2 \pm 2i}{5}}$

(f) $\frac{\sqrt{-1}}{2} = \boxed{\frac{i}{2}}$

2.

(a)
$$\begin{aligned} x + 5 &= 8 \\ x &= 8 - 5 \\ \underline{x} &\underline{= 3} \end{aligned}$$
$3 \in \underline{N}$

(b)
$$\begin{aligned} a + 3 &= 1 \\ a &= 1 - 3 \\ \underline{a} &\underline{= -2} \end{aligned}$$
$-2 \in \underline{I}$

(c)
$$\begin{aligned} x^2 + 7 &= 11 \\ x^2 &= 4 \\ \underline{x} &\underline{= \pm 2} \end{aligned}$$
$\pm 2 \in \underline{I}$

(d)
$$\begin{aligned} y^2 &= 6 \\ \underline{y} &\underline{= \pm\sqrt{6}} \end{aligned}$$
$\pm\sqrt{6} \in \underline{R}$

(e)
$$\begin{aligned} x^2 - 6x &= 0 \\ x(x - 6) &= 0 \end{aligned}$$
$\underline{x = 0}$ or $\underline{x = 6}$

$0, 6 \in \underline{W}$

(f)
$$\begin{aligned} 3x - 5 &= 0 \\ 3x &= 5 \\ \underline{x} &\underline{= \tfrac{5}{3}} \end{aligned}$$
$\frac{5}{3} \in \underline{Q}$

Exercise #2, page 4

1.

(a) pure imaginary
($a = 0$)

(b) real
($b = 0$)

(c) pure imaginary
($a = 0$)

(d) neither
($a = \frac{5}{2}$, $b = \frac{1}{2}$)

(e) pure imaginary
($a = 0$)

(f) real
($b = 0$)

Exercise #2, continued

2.

(a) $\frac{-4 + 2i}{4}$	$-1 + \frac{1}{2}i$	-1	$\frac{1}{2}$
(b) $3i$	$0 + 3i$	0	3
(c) $\frac{1 + i}{3}$	$\frac{1}{3} + \frac{1}{3}i$	$\frac{1}{3}$	$\frac{1}{3}$
(d) $-7\frac{1}{2}$	$-7\frac{1}{2} + 0i$	$-7\frac{1}{2}$	0
(e) $\frac{5 - 15i}{3}$	$\frac{5}{3} - 5i$	$\frac{5}{3}$	-5
(f) $\frac{-12 - 2i}{2}$	$-6 - i$	-6	-1

Exercise #3, page 9

1. (a) $4i + 7i = \underline{11i}$

(b) $2i - 6i = \underline{-4i}$

(c) $-5i - 2i = \underline{-7i}$

(d) $-3i + 9i = \underline{6i}$

(e) $8i - i + 3i = \underline{10i}$

(f) $2i - 6i - 3i = \underline{-7i}$

(g) $(3 + 5i) - i$
$= 3 + (5i - i)$
$= \underline{3 + 4i}$

(h) $(5 + 4i) + (6 + 2i)$
$= (5 + 6) + (4i + 2i)$
$= \underline{11 + 6i}$

(i) $(3 + 2i) + (5 - 6i)$
$= (3 + 5) + (2i - 6i)$
$= \underline{8 - 4i}$

(j) $(-6 + 4i) + (2 - 5i)$
$= (-6 + 2) + (4i - 5i)$
$= \underline{-4 - i}$

(k) $5 - (2 + 4i)$
$= 5 + (-2 - 4i)$
$= (5 - 2) - 4i$
$= \underline{3 - 4i}$

(l) $(3 + i) - (4 - i)$
$= (3 + i) + (-4 + i)$
$= (3 - 4) + (i + i)$
$= \underline{-1 + 2i}$

Exercise #3, continued

(m)
$$\begin{aligned}&(7 - 9i) - (-1 + 3i)\\ &= (7 - 9i) + (1 - 3i)\\ &= (7 + 1) + (-9i - 3i)\\ &= \underline{8 - 12i}\end{aligned}$$

(n)
$$\begin{aligned}&(5 + 3i) - 8i + (2 + i)\\ &= (5 + 2) + (3i - 8i + i)\\ &= \underline{7 - 4i}\end{aligned}$$

(o)
$$\begin{aligned}&5 + (-8 + i) + 3i\\ &= (5 - 8) + (i + 3i)\\ &= \underline{-3 + 4i}\end{aligned}$$

(p)
$$\begin{aligned}&(2 + i) - (3 - i) - (7 + 2i)\\ &= (2 + i) + (-3 + i) + (-7 - 2i)\\ &= (2 - 3 - 7) + (i + i - 2i)\\ &= \underline{-8}\end{aligned}$$

Exercise #4, page 14

1. (a) $3 + i$ (b) $-2 - 4i$ (c) $5 - 2i$

(d) $-3 + 4i$ (e) $1 - i$ (f) $-4 + i$

2. (a) $-2i \times 6i = -12i^2 = -12(-1) = \underline{12}$

(b) $-5i \times -i = 5i^2 = 5(-1) = \underline{-5}$

(c) $-3(4 + 2i) = \underline{-12 - 6i}$

(d) $2(5 - i) = \underline{10 - 2i}$

(e) $i(3 + i) = 3i + i^2 = \underline{3i - 1}$ (or $-1 + 3i$)

(f) $-5i(-2 + i) = 10i - 5i^2 = \underline{10i + 5}$ (or $5 + 10i$)

(g) $(4 + 2i)(4 - 2i) = (4)^2 - (2i)^2 = 16 - 4i^2 = 16 + 4 = \underline{20}$

(h) $(7 + 6i)(8 - 3i) = 56 + 48i - 21i - 18i^2 = 56 + 27i + 18 = \underline{74 + 27i}$

(i) $(6 - 5i)(-4 + 3i) = -24 + 20i + 18i - 15i^2 = -24 + 38i + 15 = \underline{-9 + 38i}$

(j) $(5 - 6i)(2 - 3i) = 10 - 12i - 15i + 18i^2 = 10 - 27i - 18 = \underline{-8 - 27i}$

(k) $(3 + 7i)(3 - 7i) = (3)^2 - (7i)^2 = 9 - 49i^2 = 9 + 49 = \underline{58}$

Exercise #5, page 18

1. (a) $\dfrac{-20i}{-4i} = \boxed{5}$

(b) $\dfrac{3}{2 - 5i} = \dfrac{3}{2 - 5i} \times \dfrac{2 + 5i}{2 + 5i} = \dfrac{6 + 15i}{4 - 25i^2} = \dfrac{6 + 15i}{29} = \boxed{\dfrac{6}{29} + \dfrac{15}{29}i}$

Exercise #5, continued

(c) $\frac{2+i}{-1-i} = \frac{2+i}{-1-i} \times \frac{-1+i}{-1+i} = \frac{-2-i+2i+i^2}{1-i^2} = \frac{-3+i}{2} = \enclose{circle}{\frac{-3}{2} + \frac{1}{2}i}$

(d) $\frac{-5}{3i} = \frac{-5}{3i} \times \frac{i}{i} = \frac{-5i}{3i^2} = \frac{-5i}{-3} = \enclose{circle}{\frac{5}{3}}i$

(e) $\frac{2+i}{1+3i} = \frac{2+i}{1+3i} \times \frac{1-3i}{1-3i} = \frac{2+i-6i-3i^2}{1-9i^2} = \frac{5-5i}{10} = \enclose{circle}{\frac{1}{2} - \frac{1}{2}i}$

Exercise #6, page 19

1. (a) $i^{15} = (i^2)^7 \times i = (-1)^{\overset{\swarrow\text{ODD NUMBER}}{7}} \times i = -1 \times i = \underline{-i}$

(b) $i^8 = (i^2)^4 = (-1)^{\overset{\swarrow\text{EVEN NUMBER}}{4}} = \underline{1}$

(c) $2i - 3i^3 = 2i - 3(i^2)(i) = 2i - 3(-1)(i) = 2i + 3i = \underline{5i}$

(d) $6i^4 - 2i^5 = 6(i^2)^2 - 2(i^2)^2(i) = 6(-1)^2 - 2(-1)^2(i) = \underline{6 - 2i}$

(e) $i \times 3i \times -5i = -15i^3 = -15(i^2)(i) = -15(-1)(i) = \underline{15i}$

Exercise #7, page 20

1. (a)
$$\begin{aligned} x^2 - 3 &= 0 \\ x^2 &= 3 \\ \underline{x} &\underline{= \pm\sqrt{3}} \end{aligned}$$

(b)
$$\begin{aligned} x^2 + 3 &= 0 \\ x^2 &= -3 \\ x &= \pm\sqrt{-3} \\ \underline{x} &\underline{= \pm\sqrt{3}i} \end{aligned}$$

(c) $x^2 + x - 3 = 0$

$a = 1,\ b = 1,\ c = -3$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1 + 12}}{2}$$

$$= \boxed{\frac{-1 \pm \sqrt{13}}{2}}$$

(d) $x^2 - 3x + 4 = 0$

$a = 1,\ b = -3,\ c = 4$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{9 - 16}}{2}$$

$$= \frac{3 \pm \sqrt{-7}}{2}$$

$$= \boxed{\frac{3 \pm \sqrt{7}i}{2}}$$

Exercise #7, continued

(e) $x^2 - 6x + 13 = 0$

$a = 1,\ b = -6,\ c = 13$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(13)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$x = \frac{6 \pm \sqrt{-16}}{2}$$

$$x = \frac{6 \pm 4i}{2}$$

$$x = \underline{3 \pm 2i}$$

(f) $2x^2 - 5x - 3 = 0$

$(2x + 1)(x - 3) = 0$

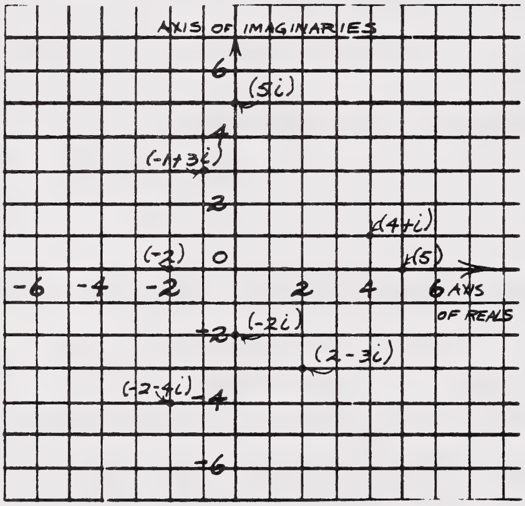
If $2x + 1 = 0$ If $x - 3 = 0$

$2x = -1$ $\boxed{x = 3}$

$\boxed{x = \frac{-1}{2}}$

Exercise #8, page 23

1.



2. (a) $-8 + 5i$

(b) $-4 + 2i$

(c) $8i$

(d) $5 + 6i$

(e) $9 + 6i$

(f) 4

(g) $7 - 6i$

(h) $2 - 9i$

(i) $-3i$

(j) $-3 - 8i$

(k) $-5 - 3i$

(l) -8

Exercise #9, page 38

1. (a) add
(3y and -3y are additive inverses.)

(b) add
(2x and -2x are additive inverses.)

(c) subtract
(4m and 4m are identical terms.)

(d) subtract
(b and b are identical terms.)

Exercise #9, continued

2. (a)

$$\begin{array}{rl} 11x - y = 42 & (1)\\ 3x + y = 0 & (2)\\ \hline 14x = 42 & \text{(add)}\\ \boxed{x = 3} & \end{array}$$

Substitute 3 for x in (1).

$$\begin{array}{r} 11(3) - y = 42\\ 33 - y = 42\\ -y = 9\\ \boxed{y = -9} \end{array}$$

The solution set is $\underline{\{(3, -9)\}}$.

(b)

$$\begin{array}{rl} 4x + 2y = 1 & (1)\\ 4x + 3y = -5 & (2)\\ \hline -y = 6 & \text{(subtract)}\\ \boxed{y = -6} & \end{array}$$

Substitute -6 for y in (1).

$$\begin{array}{r} 4x + 2(-6) = 1\\ 4x - 12 = 1\\ 4x = 13\\ \boxed{x = 3\frac{1}{4}} \end{array}$$

The solution set is $\underline{\{(3\frac{1}{4}, -6)\}}$.

Exercise #10, page 39

1. (a) We could eliminate the variable x if we multiplied equation (2) by 2 and then subtracted the two equations.

(b) We could eliminate the y's if we multiplied equation (1) by 3, multiplied equation (2) by 2 and added the two equations.

(c) We could eliminate the y's if we multiplied equation (1) by 4, multiplied equation (2) by 5 and subtracted the two equations.

2. (a)

$$\begin{array}{rl} 6x + 15y = -3 & (1) \times 3\\ 6x + 14y = 4 & (2) \times 2\\ \hline \boxed{y = -7} & \text{(subtract)} \end{array}$$

Substitute -7 for y in (1).

$$\begin{array}{r} 2x + 5(-7) = -1\\ 2x - 35 = -1\\ 2x = 34\\ \boxed{x = 17} \end{array}$$

The solution set is $\{(17, -7)\}$.

(b)

$$\begin{array}{rl} 5a + 15b = 50 & (1) \times 5\\ 5a + 7b = 2 & (2)\\ \hline 8b = 48 & \text{(subtract)}\\ \boxed{b = 6} & \end{array}$$

Substitute 6 for b in (1).

$$\begin{array}{r} a + 3(6) = 10\\ a + 18 = 10\\ \boxed{a = -8} \end{array}$$

The solution set is $\{(-8, 6)\}$.

Note: In question #2 above, your solutions will not parallel those given if you chose to eliminate different variables. You still should get the same result though.

Exercise #11, page 42

1. (a)

$$\begin{aligned} 3x + 2y &= 29 \quad (1)\\ y - x &= -8 \quad (2) \end{aligned}$$

From (2), $y - x = -8$

$$y = x - 8 \quad (3)$$

Substitute $x - 8$ for y in (1).

$$\begin{aligned} 3x + 2(x - 8) &= 29\\ 3x + 2x - 16 &= 29\\ 5x - 16 &= 29\\ 5x &= 45\\ x &= 9 \end{aligned}$$

Substitute 9 for x in (3).

$$\begin{aligned} y &= x - 8\\ y &= 9 - 8\\ y &= 1 \end{aligned}$$

Solution set is $\{(9,1)\}$.

(b)

$$\begin{aligned} 3x - 4y &= 0 \quad (1)\\ 2x + y &= 33 \quad (2) \end{aligned}$$

From (2), $2x + y = 33$

$$y = 33 - 2x \quad (3)$$

Substitute $33 - 2x$ for y in (1).

$$\begin{aligned} 3x - 4(33 - 2x) &= 0\\ 3x - 132 + 8x &= 0\\ 11x &= 132\\ \therefore, \; x &= 12 \end{aligned}$$

Substitute 12 for x in (3).

$$\begin{aligned} y &= 33 - 2x\\ y &= 33 - 2(12)\\ y &= 9 \end{aligned}$$

Solution set is $\{(12,9)\}$.

Note: In parts (a) and (b) above, your solutions will not parallel those given if you isolated different variables or substituted in the other equation. You should still get the same result though.

Exercise #12, page 45

1. (a) infinite number of solutions; dependent
(Both equations can be put in the form $y = x$.)

(b) One real solution; independent.

(c) infinite number of solutions; dependent
(Both equations can be put in the form $y = 3x + 1$.)

(d) no real solution; inconsistent
(The two equations differ only in their constant terms.)

(e) one real solution; independent

(f) no real solution; inconsistent
(The two equations differ only in the constant terms.)

Exercise #13, page 50

1. (a) straight line (form $y = ax + b$)
hyperbola (form $ax^2 - by^2 = c$)

(b) straight line (form $y = ax + b$)
straight line (form $y = ax + b$)

(c) parabola (form $y = ax^2 + bx + c$)
straight line (form $y = ax + b$)

(d) ellipse (form $ax^2 + by^2 = c$)
straight line (form $y = ax + b$)

End of Lesson I

Lesson J

GEOMETRY

Make sure you have a metric ruler and a protractor before you attempt this lesson. In this lesson, you are asked to make a number of measurements and do some constructions in metric units.

In a single lesson in geometry, we are unable to present the network of postulates and theorems that are outlined in the geometry lessons of Mathematics 10 and 20. In Lesson J, you will find only a summary of the basic geometric facts that you will need for Mathematics 30.

Geometry had its beginnings either in Egypt or Babylonia. This early geometry consisted of observation and measurement. People used geometry in finding volumes and areas, in doing simple surveying, in planning buildings, and in decoration.

The best known of the Greek geometers was Euclid who wrote his famous Elements in about 300 B.C. This geometry text of Euclid's contained the first systematic, orderly, and logical arrangement of geometry. It has been held in the highest respect from the time of its writing to the present.

Topic One: Basic Geometric Terms

A. Undefined Terms

In geometry, we are concerned with such ideas as "point", "line", and "plane". If you were asked to define a "point", you might say that it is the intersection of two lines. If you were pressed further and asked to define a "line" you might say that it is a set of points. By now you have probably noticed that these definitions are dependent on each other. We allow such terms to remain undefined, since any attempt to explain them results in either the kind of circular reasoning we have just illustrated or the use of terms more complicated than the terms being explained.

Thus, in geometry, we consider "point", "line" and "plane" to be undefined terms. Then it is possible to define other words in terms of these three words. The rest of our geometric terms can be built from this framework of undefined terms.

I. Point

The simplest figure that can be drawn in geometry is a "point". All other geometric figures are made up of points.

A point, as we mean it in geometry, has location but no length, width, nor thickness. That is, a point has location but no size.

A point can be represented by a dot on a piece of paper. To be very technical, the dot you make on paper to locate position is not a point at all, since it has certain length, width, and thickness. But, of course, we have to make visible marks to represent points. It is the custom to name points with capital letters.

EXAMPLE:

.B

.R

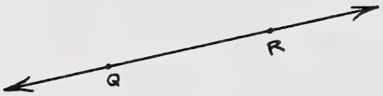
Point B and point R have been located on the left.

II. Line

A geometric line has only one dimension; length. Unless otherwise stated, the word "line" means "straight line". A line can be thought of as an infinite set of points. Between any two points in a line there is another point. Also, there are no gaps between the points in a line. A line has no endpoints but continues indefinitely in two directions.

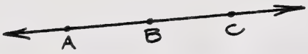
We can represent a line by drawing a stroke with a ruler and placing arrowheads at both ends of it. The arrowheads indicate that the line has no ends. A line is named by referring to any two points that lie in it. A double-headed arrow is then placed above these two capital letters that are used to name the line.

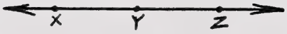
EXAMPLE:

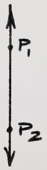


We can name this figure $\overleftrightarrow{QR}$ (read "line QR") or $\overleftrightarrow{RQ}$ (read "line RQ").

Name each of the lines below. (Remember that any two points in a line can be used to name it.)







$\overleftrightarrow{AB}$ ______

______

______

In the space provided below, draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ that intersect at point M. (Remember to put arrowheads at both ends of the lines.)

Points that lie in the same straight line are called COLLINEAR POINTS while points that do not lie in the same straight line are called NON-COLLINEAR POINTS. In the diagram you drew above, note that points A, M, and B are collinear points, while points A, M and C are non-collinear points. Are points D, M, and B collinear or non-collinear?

____

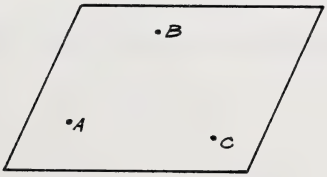
Are points D, M, and C collinear or non-collinear? __________________________

III. Plane

In space, there are certain sets of points which we will call "planes". Familiar objects with flat surfaces such as table tops or mirrors suggest planes, but these are only physical models of planes. A plane is a mathematical abstraction.

A plane has infinite length and width but no thickness. When we draw a picture to represent a plane, we draw a 4-sided figure. Any three non-collinear points determine a plane, so we use three points in a plane to name it.

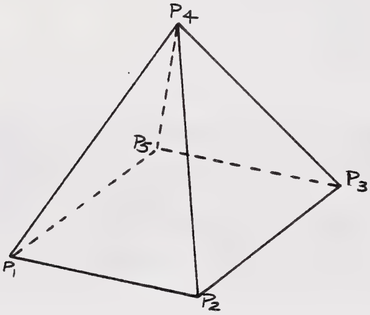
EXAMPLE:



This is "plane ABC".

Any figure that can be drawn in a single plane is called a PLANE FIGURE. For example, lines, angles, triangles, and circles are all plane figures. On the other hand, such figures as cubes, cones, and cylinders have a third dimension (thickness) and hence are called SPACE FIGURES.

Points that lie in the same plane are called COPLANAR POINTS while points that do not lie in the same plane are called NON-COPLANAR POINTS.
EXAMPLE:



Note that P_1 P_4, P_5, are coplanar points while P_1, P_4, P_5, and P_2 are not.

Name another three coplanar points.

____

Name another four non-coplanar points.

____

Are P_1, P_2, P_3, and P_5 coplanar points? __________

B. Definitions of Plane Geometric Figures

By building upon the undefined terms "point", "line" and "plane", we can define other geometric terms. A good definition

(1) names the term to be defined.

(2) places it in the smallest class to which it belongs.

(3) gives the definite characteristics which distinguish it from other members of that class.

(4) is reversible.

In the sections that follow, we will define some of the more important geometric plane figures.

I. Segment

A SEGMENT is part of a line consisting of any two distinct points and all the points between.

Notice that in our definition of "segment" we have

(1) mentioned the term "segment" which we are defining.

(2) placed segment in the class of all points in a line.

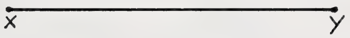
(3) distinguished a segment from other subsets of a line by stating that it consists of any two distinct points and all the points between.

(4) made the subject and predicate interchangeable. (i.e. the definition could also be written in the form "a part of a line consisting of any two distinct points and all the points between is a segment.")

Like a line, a segment has no width nor thickness. But, unlike a line, it has definite length rather than infinite length. The length of a segment can be measured with a ruler.

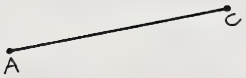
We can represent a segment by making two dots and joining them. A segment is named by referring to its two endpoints (in either order.) A bar is then placed above these two capital letters that are used to name the segment.

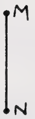
EXAMPLE:

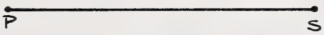


We can name this figure $\overline{XY}$ (read "segment XY") or $\overline{YX}$ (read "segment YX").

Name each segment below. (Remember that a segment is always named by its two endpoints and a bar is placed over the two letters.)







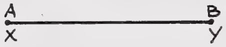
$\overline{AC}$ ______ ______

The lengths of all segments can be measured. When we wish to refer to the measure of a segment, we omit the bar above the two letters that name the segment. For example, XY refers to the measure of $\overline{XY}$. If we measure $\overline{XY}$ on page 5, we find that XY = 5.1 cm. (In other words, the measure of segment XY is 5.1 cm.

Give the measures of the other three segments at the bottom of page 5 to the nearer tenth centimetre.

AC = __________ __________ __________

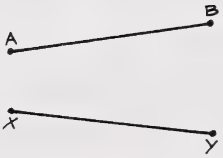
Two segments are said to be EQUAL if they consist of exactly the same set of points.
EXAMPLE:



$\overline{AB} = \overline{XY}$ since A and X name the same point and B and Y name the same point.

Two segments are said to be CONGRUENT if they have the same measure. We use the symbol ≅ to mean "is congruent to."
EXAMPLE:

$\overline{AB} \cong \overline{XY}$



This is read, "segment AB is congruent to segment XY."
These segments are congruent since AB = XY = 3.2 cm. (i.e. $\overline{AB}$ and $\overline{XY}$ are congruent since they have the same measure.)

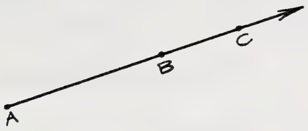
II. Ray

A RAY is part of a line consisting of a given point and all the points on one side of it.

Like a line, a ray has no width nor thickness, but has infinite length. But, unlike a line, it extends indefinitely in only one direction.

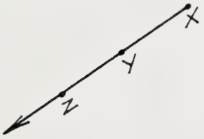
We can represent a ray by drawing a line and placing an arrowhead at one end of the line. The arrowhead indicates that the ray extends indefinitely in one direction. A ray is named by stating the endpoint first and then giving one other point on the ray. An arrow pointing to the right is then placed above these two letters that are used to name the ray.

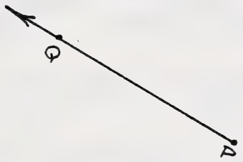
EXAMPLE:

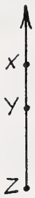


We name this figure $\overrightarrow{AB}$ (read "ray AB") or $\overrightarrow{AC}$ (read "ray AC").

Name each ray below. (Remember that a ray is always named by giving its endpoint first and then one other point. An arrow pointing to the right is placed over the two letters.)







$\overrightarrow{XZ}$ ________

____

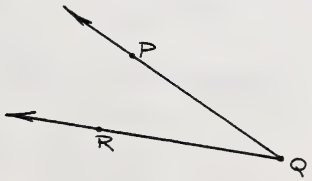
____

III. Angle

An ANGLE is a set of points formed by two rays having the same endpoint.

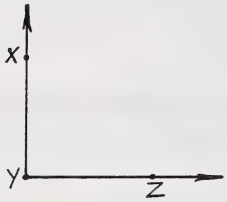
We may represent an angle by drawing two rays that begin at the same point and go off indefinitely in two directions. The common endpoint of the rays is called the VERTEX of the angle.

EXAMPLE:



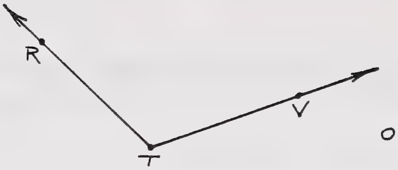
This angle is composed of $\overrightarrow{QP}$ and $\overrightarrow{QR}$ that have the common endpoint Q. Q is called the vertex of the angle.

For the angles below, name the rays of which they are composed and the vertex.



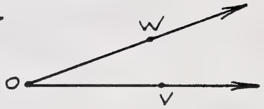
Rays: $\overrightarrow{YX}$, ____

Vertex: ____



Rays: ____, ____

Vertex: ____



Rays: ____, ____

Vertex: ____

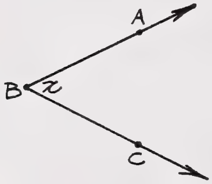
When writing about angles, we often use the symbol "∠" as a short form of the word "angle". There are three common ways of naming an angle:

(1) by a capital letter at its vertex.

(2) by a small letter or number within the angle.

(3) by three capital letters - a point on one ray, the vertex, and a point on the other ray. (The middle letter given must always be the vertex.)

EXAMPLE:



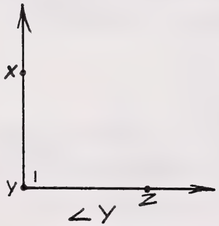
We could name this angle as follows:

(1) ∠B

(2) ∠x

(3) ∠ABC
VERTEX

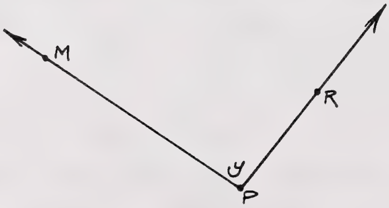
Name each angle below by all three methods of naming angles.



∠Y

∠1

∠XYZ



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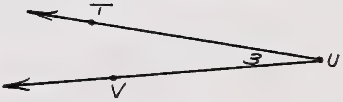


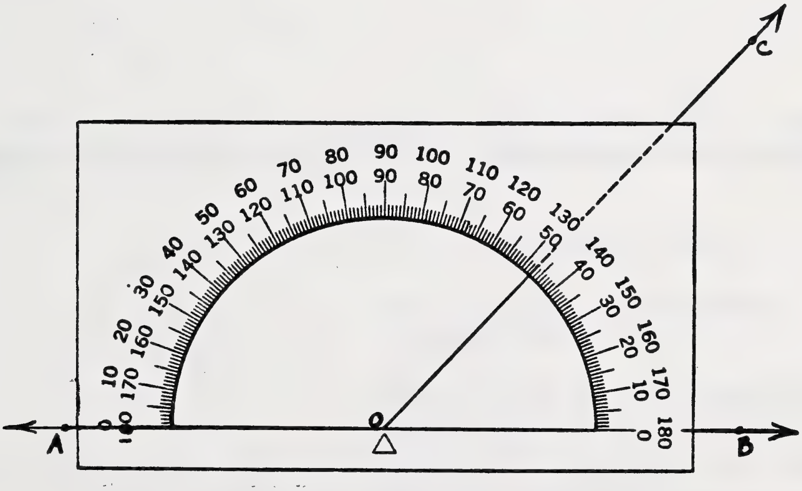
____

____

____

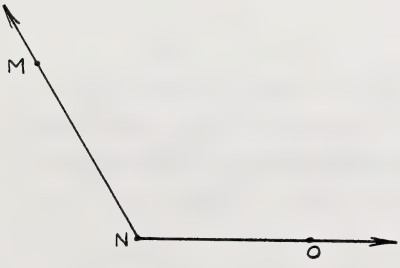
The measure of an angle is given in DEGREES. The symbol m∠ABC represents the "measure of angle ABC". We use a PROTRACTOR to measure angles. There are 180 equally spaced divisions around the edge of a protractor. Each of these divisions represents a measure of one degree. (1°) DEGREE SYMBOL A protractor has an inner and outer scale. The inner scale can be used to measure angles that open to the right and the outer scale can be used to measure angles that open to the left.

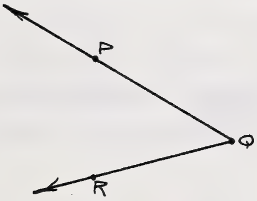
EXAMPLE:

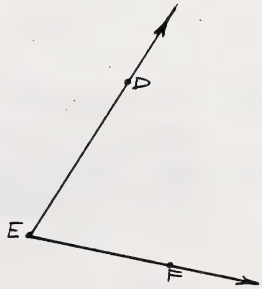


∠AOC and ∠COB are shown above. A protractor has been placed with its centre at the vertex of these angles. The outer scale of the protractor tells us that m∠AOC = 133°. The inner scale of the protractor tells us that m∠COB = 47°.

State the measure of each of the angles below. (Extend the rays of the angles if this helps you read the values more accurately.







m∠ ____ = ____°.

________ = ____

________ = ____

Angles may be classified according to their measures.

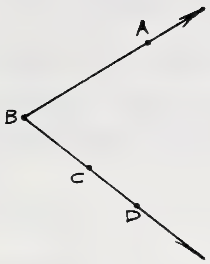
1. A RIGHT ANGLE has a measure of 90°. Draw ∠MNO below which is a right angle. (Remember to put an arrowhead at the end of each ray.)

2. An ACUTE ANGLE has a measure less than 90°. Draw ∠YZV below which is an acute angle. What is the measure of this angle? ________

3. An OBTUSE ANGLE has a measure between 90° and 180°. Draw ∠FGH below which is an obtuse angle. What is the measure of this angle? __________

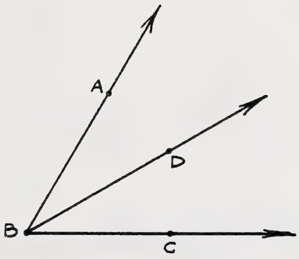
4. A STRAIGHT ANGLE has a measure of 180°. Draw ∠XYZ below which is a straight angle. The rays of a straight angle are called OPPOSITE RAYS because they form a straight line. Name the two opposite rays in ∠XYZ. _____, _____

Two angles are said to be EQUAL if they consist of the same set of points.
EXAMPLE:



$\angle ABC = \angle ABD$ since they are names for the same angle.

Two angles are said to be CONGRUENT if they have the same measure.
EXAMPLE:



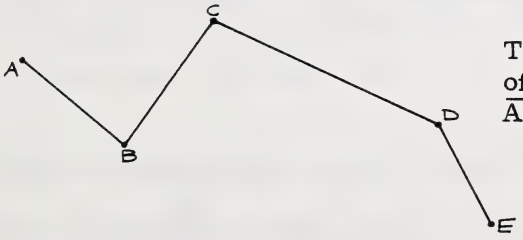
$\angle ABD \cong \angle DBC$

This is read, "angle ABD is congruent to angle DBC".

These angles are congruent since $m\angle ABD = m\angle DBC = 30°$. (i.e. these angles are congruent since they have the same measure.)

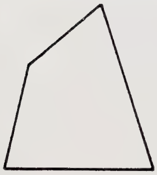
IV. Polygon

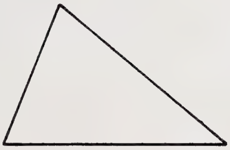
When several segments are joined together in succession, we call the result a BROKEN LINE. The figure below is a broken line.

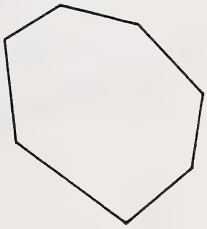


This broken line is composed of 4 segments. They are $\overline{AB}$, $\overline{BC}$, ______, and ______.

When a broken line is closed to bound a portion of a plane, it is called a POLYGON. The three figures below are all polygons.



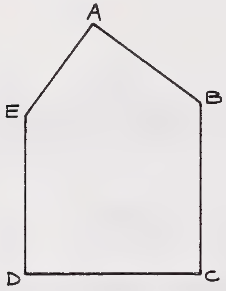




A POLYGON is a closed broken line.

Each line segment which is part of a polygon is called a SIDE of the polygon. What is the least number of sides that a polygon can have? ____ Each point where two line segments meet is called a VERTEX of the polygon. A polygon has the same number of vertices as it has sides. We usually name a polygon by listing the letters at the vertices in a clockwise fashion, starting with the letter closest to the beginning of the alphabet.

EXAMPLE:



This is polygon ABCDE.

How may sides has this polygon? ______

How many vertices? ______

Draw a 6-sided polygon MNOPQR in the box below and then name its vertices and sides.

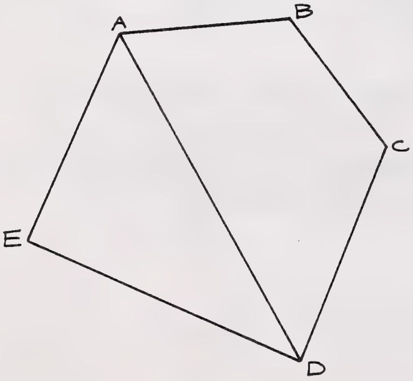
Its vertices are M, ____________________

Its sides are $\overline{MN}$, ____________________

____

When we join two vertices of a polygon that are not adjacent (i.e. they do not lie right next to each other), we obtain a DIAGONAL of the polygon.

EXAMPLE:



$\overline{AD}$ is a diagonal of polygon ABCDE.

Draw in the other 4 diagonals. Name these diagonals.

_____, _____, _____, _____

V. Circle

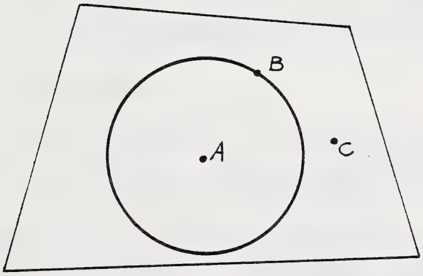
A CIRCLE is the set of all points in a plane that are a fixed distance from a given point in the plane.

The given point is called the CENTER of the circle.

Although we must refer to the "center" of a circle when defining a circle, the center does not belong to the set of points that make up a circle.

A circle is named by naming the point that is its center.

EXAMPLE:



This is circle A which lies in plane plane ABC.

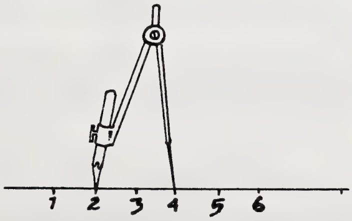
Is point A a subset of circle A? ______

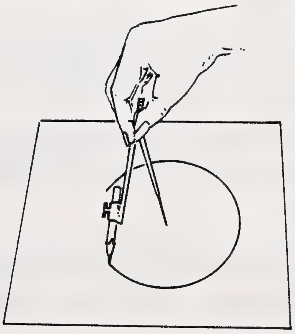
Is point B a subset of circle A? ______

Is point C a subset of circle A? ______

We use a compass to construct circles. The sharp point of the compass represents the center of the circle and the pencil point traces the path of points that are equidistant from the center.

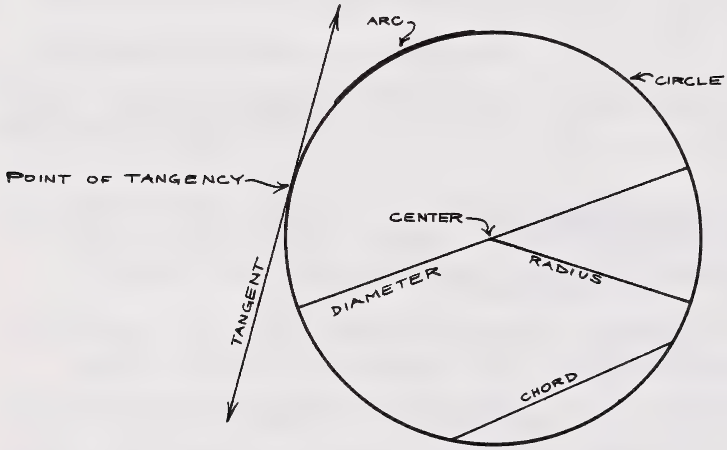
In order to draw a circle of a specified radius, open the legs of the compass to the desired length. For example, to draw a circle with a radius of 2 cm, put the legs of the compass along a ruler and measure off 2 cm. Then, keeping this measured radius, place the tip of the compass on a piece of paper and rotate the pencil point through one complete turn.





In the space provided on the left, draw circle P with a radius of 2 cm.

There are certain terms associated with circles whose meanings you should learn. Study the diagram below and the definitions on the next five pages.

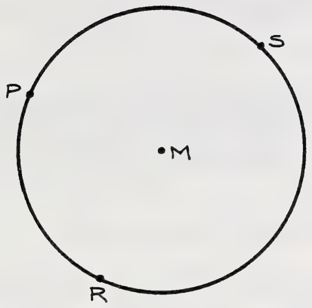


1. Radius

A RADIUS of a circle is a line segment having the center of the circle as one of its endpoints and a point on the circle as the other endpoint.

Since a circle is composed of an unlimited number of points, there are an unlimited number of radii (plural of radius) associated with a given circle.

Since a radius is a line segment, it has a definite length which can be measured. All the radii of a given circle have the same measure.



Name this circle. ____________________

Name three points that belong to this circle. ________________

Draw three radii by joining the center of the circle to points P, R, and S.

Name these three radii. $\overline{MP}$, ______________

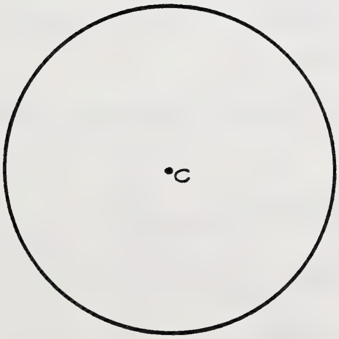
Give the measure of these radii to the nearer tenth centimetre. ______________

2. Diameter

A DIAMETER of a circle is a line segment passing through the center of the circle and having both endpoints on the circle.

There are an unlimited number of diameters associated with a given circle because an unlimited number of points lie in the circle.

Since a diameter is a line segment, we can find its measure. All the diameters of a given circle have the same measure. The measure of a diameter of a circle will always be twice the measure of a radius.



Draw $\overline{CB}$ which is a radius of circle C.

What is the measure of $\overline{CB}$ in cm (to one-decimal place)?

____

Draw $\overline{AD}$ which is a diameter of circle C. ($\overline{AD}$ must pass through the center C.)

What is the measure of $\overline{AD}$ in cm (to one decimal place)?

____

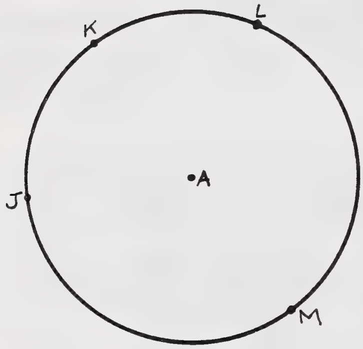
Is the measure of $\overline{AD}$ twice that of $\overline{CB}$?

____

3. Chord

A CHORD of a circle is a line segment whose endpoints lie on the circle.

There are an unlimited number of chords associated with a given circle. Each of these chords has a measure that depends upon which points on the circle are its endpoints. Two chords of a given circle are congruent if they have the same measure.
NOTE: Any diameter of a circle is also a chord of that circle.



Join points J, K, L, and M in such a way as to form six chords of circle A.

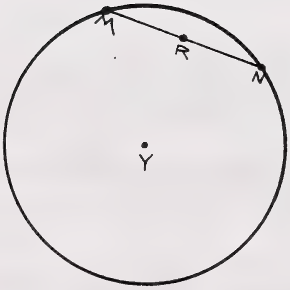
Name two pairs of congruent chords.

$\overline{JK} \cong$ _____

_____ $\cong$ _____

Name a chord which is also a diameter of the circle. _____

If we join the midpoint of a chord to the center of the circle, a right angle is always formed.



$\overline{MN}$ is a chord of circle Y. R is the midpoint of $\overline{MN}$ (i.e. R is half-way between M and N.)

Join points Y and R and then measure $\angle YRN$.

$m\angle YRN =$ _____°

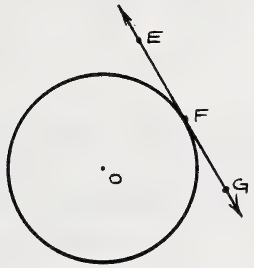
In other words, $\angle YRN$ is a

__________ angle.

4. Tangent

A line is TANGENT to a circle if it touches the circle at exactly one point.

The point of intersection of the tangent with the circle is called the POINT OF TANGENCY. If a radius is drawn to the point of tangency, it forms a right angle with the tangent.



$\overleftrightarrow{EG}$ is tangent to circle O.

What is the point of tangency? ________

Join points O and F in the diagram.

Measure ∠OFG.

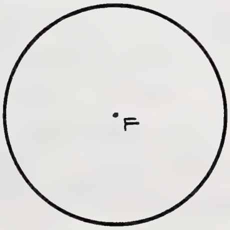
$m\angle OFG =$ ___°.

In other words, ∠OFG is a

__________ angle.

If a line is not tangent to a circle, it either has no points in common with the circle or it has two points in common with the circle.

Draw $\overleftrightarrow{AB}$ which has no points in common with circle M.



Draw $\overleftrightarrow{XY}$ which has the two points, B and C in common with circle F.

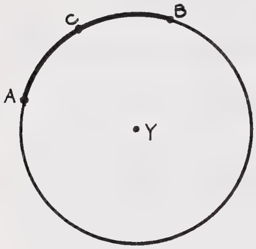
What is the greatest number of points that a line can have in common with a circle? ______________

5. Arc

> An ARC is a subset of a circle that is composed of two points in the circle and all the points between.

An arc may be named by referring to its two endpoints and one other point. (The "other point" is listed in the middle.) A curved line is placed over the three letters used to name the arc.

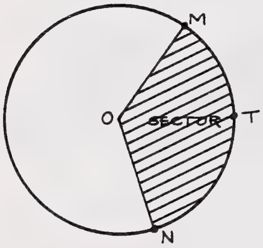
EXAMPLE:



An arc with endpoints A and B that includes point C has been outlined in circle Y on the left. We can call this arc, $\overset{\frown}{ACB}$ (which is read, "arc ACB").

If we join the endpoints of an arc to the center of a circle, we form a SECTOR of the circle. The angle that is formed is called a CENTRAL ANGLE.

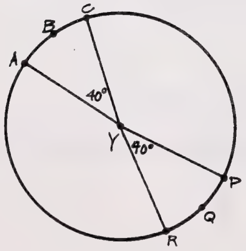
EXAMPLE:



The endpoints M and N of $\overset{\frown}{MTN}$ have been joined to the center of circle O. The sector that is formed is shaded on the diagram. $\angle MON$ is called a central angle.

Two arcs are congruent if their central angles are congruent. Also, if two arcs are congruent, the chords that are formed by joining their endpoints are also congruent.

EXAMPLE:



$\overset{\frown}{ABC}$ and $\overset{\frown}{PQR}$ determine two sectors with central angles with measures of 40° each. Since these central angles are congruent, the arcs must be congruent.

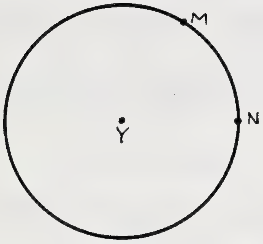
$$\text{i.e.}\quad \overset{\frown}{ABC} \cong \overset{\frown}{PQR}$$

Join points A and C and points P and R to form chord AC and chord PR. Note that

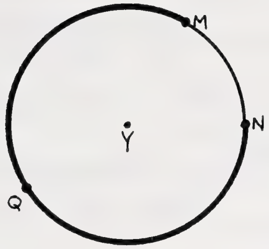
$$\overline{AC} \cong \overline{PR}$$

Any two points in a circle determine two arcs. If these two arcs are not congruent, the larger arc is called the MAJOR ARC and the smaller arc is called the MINOR ARC.

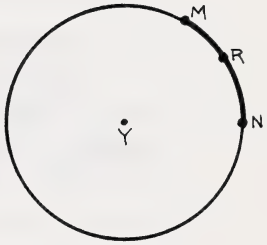
EXAMPLE:



Points M and N divide circle Y into two non-congruent.



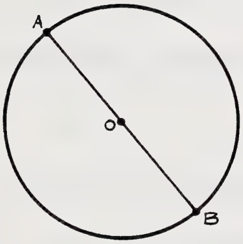
$\overset{\frown}{MQN}$ is a major arc.



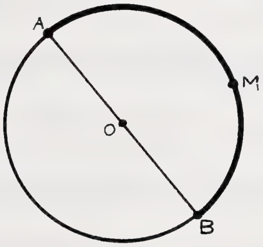
$\overset{\frown}{MRN}$ is a minor arc.

If two points divide a circle into two congruent arcs, then each of these arcs is called a SEMICIRCLE. The endpoints of a diameter of a circle determine two semicircles.

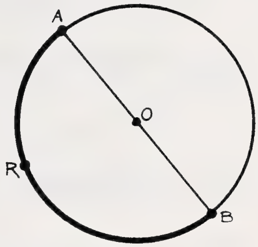
EXAMPLE:



$\overline{AB}$ is a diameter of circle O since A and B are on the circle and $\overline{AB}$ passes through the center O.



$\overset{\frown}{AMB}$ is a semicircle.



$\overset{\frown}{ARB}$ is a semicircle.

EXERCISE - Basic Geometric Terms

1. Fill in the blanks.

(a) Points, ______________, and ________________ are subsets of a line.

(b) A ________________ has location but no length, width nor thickness.

(c) ________________ and ______________ have infinite length, but no width nor thickness.

(d) ________________ points are points that do not lie in the same straight line.

(e) A wall of a room is a physical model of a ________________.

(f) A ________________ consists of two points in a line and all the points between.

(g) A ________________ consists of a given point in a line and all the points on one side of it.

(h) An angle is composed of two ______________ that have a common ______________.

(i) When you name an angle by referring to three points, you must name the ______________ in the middle.

(j) A segment is named by referring to its two ________________.

(k) A _______________ is used to measure an angle.

(l) A(n) _____________ angle has a measure between 90° and 180°.

(m) A _______________ angle has a measure of 90°.

(n) Points which lie in the same plane are called ____________________ points.

(o) A _______________ is a closed broken line.

(p) Two angles are ______________ if they have the same measure.

(q) Any point where two sides of a polygon meet is called a __________ of the polygon.

(r) A circle is named by referring to its ____________________.

(s) The radius of a circle is a _________________ having the ________ of the circle as one of its endpoints and a point on the circle as the other endpoint.

(t) A _____________ is a segment passing through the center of the circle and having both endpoints on the circle.

(u) A _____________ of a circle is a segment whose endpoints lie on the circle.

(v) A _____________ to a circle touches the circle at exactly one point.

(w) In a circle, chords of congruent minor arcs are _________________.

(x) The symbol AB represents the _________ of segment AB.

(y) If two points divide a circle into two congruent arcs, each arc is called a _________________________.

(z) Any two points in a circle determine two arcs. The larger of these two arcs is called a ___________ arc.

2. Write each of the following in symbols.

(a) ray CD __________

(b) line MN __________

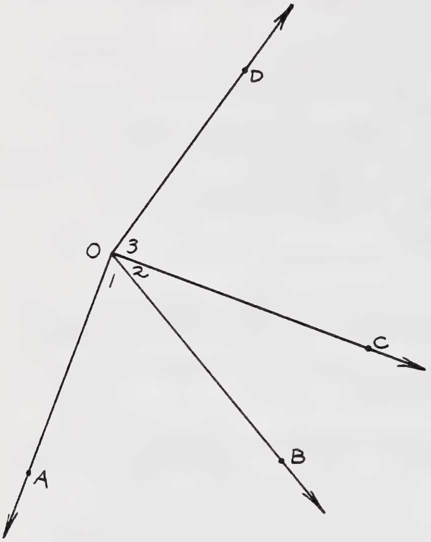
(c) segment RS __________

(d) angle MNO __________

(e) arc XYZ __________

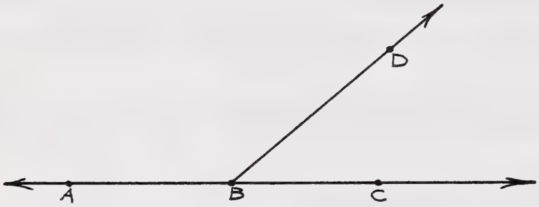
(f) the measure of angle R __________

3. In the figure below, name the six angles by a number or by three letters, whichever is simpler and clearly identifies the angle. Then state the measure of each angle and classify it as being acute, obtuse, or right.



Angle	Measure	Classification
∠1	60°	acute
∠DOB		

4.



(a) Name a line. ______

(b) Name three different segments that are subsets of this line.

______, ______, ______

(c) Name three different rays with an endpoint of B. ______, ______, ______

(d) Name an acute angle. ______ Name a straight angle. ______

Name an obtuse angle. ______

(e) Name three collinear points. ______, ______, ______

(f) Name the endpoint of $\overrightarrow{CA}$. ______ Name the endpoints of $\overline{DB}$. ______, ______

(g) Name the vertex of ∠DBC. ______

(h) Name the rays that form ∠DBA. ______, ______

5. Make drawings to represent each of the following geometric figures. (Use a straight edge to draw segments, lines and rays, and a compass to draw arcs and circles.)

(a) $\overline{XY}$ which measures 4.5 cm

(b) $\angle ABC$ which measures 65°

(c) Plane RST

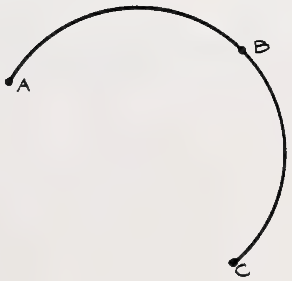
(d) $\overleftrightarrow{XY}$ and $\overleftrightarrow{MN}$ that intersect at P so that $m\angle XPN = 90°$

(e) $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ that are opposite rays

(f) Polygon VWXY with diagonal $\overline{WY}$ drawn

(g) Circle S with chord MN drawn

(h) $\overset{\frown}{ABC}$ which is a major arc



(i) $\overset{\frown}{XYZ}$ which is a minor arc

(j) Circle C with central angle BCA

(k) Circle M with diameter $\overline{XY}$ which measures 4 cm

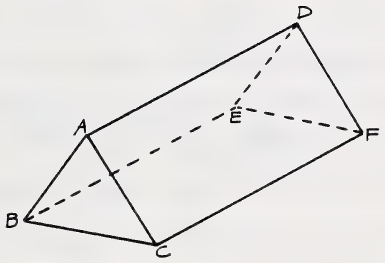
(l) Circle O and tangent $\overleftrightarrow{MN}$. Point of tangency is Y. Draw radius $\overline{OY}$.

(m) A broken line composed of $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$.

(n) $\overset{\frown}{PQR}$ which is a semicircle.

6. From any point P, draw two rays forming an angle of 40°. On each ray, locate a point 4 cm from P. Letter these points M and N. Draw $\overline{MN}$. Measure ∠PMN and ∠PNM.

7.



(a) Name one other point that is coplanar with points A, B, and E. ______

(b) Name one other point that is coplanar with points B, E, and F. ______

(c) Name an angle in plane ABC. ________

(d) Name a segment in plane DEF. ________

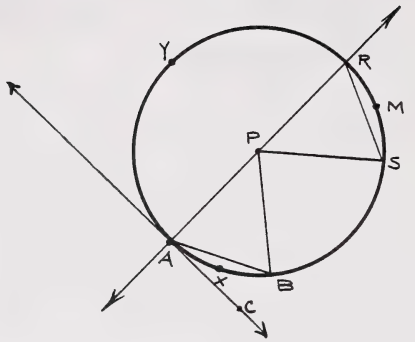
(e) Name a segment that is common to both plane ABC and plane ABE.

____

(f) Name the four segments that make up polygon ACFD.

_____, _____, _____, _____

8.



(a) Name the circle. __________

(b) Name a diameter of the circle. __________ The measure of this diameter is ________ cm (to one decimal place).

(c) Name any line segments that appear in the diagram that are radii of the circle. $\overline{PR}$, __________ Each of these radii measures ______ cm.

(d) Name three line segments that appear in the diagram that are chords of the circle. __________

(e) Name all points that are subsets of the circle. ________________

(f) Name two semicircles. __________, __________

(g) Name two congruent acute central angles. __________, __________

(h) Name two congruent minor arcs. __________

(i) Name two congruent chords. __________, __________

(j) Name a line which is tangent to the circle. __________

What is the point of tangency? ______

Name the radius that forms a right angle with this tangent. ________

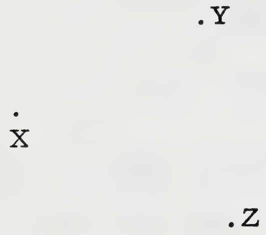
(k) Name a line which is not tangent to the circle. __________

How may points has this line in common with the circle? ________

Topic Two: Triangles

A TRIANGLE is a three-sided polygon that can be formed by joining three non-collinear points. Each of these points is called a VERTEX of the triangle. A triangle is named by using the symbol △ which is read "triangle" and listing the vertices in any order.

EXAMPLE:



Join the 3 non-collinear points X, Y, and Z to form a triangle.

We can name this triangle △XYZ. The three vertices of this triangle are

_____, _____, and _____.

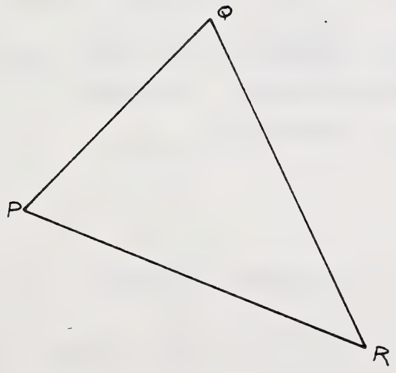
A. Parts of a Triangle

Every triangle is made up of three sides that are segments. These sides have definite lengths that can be measured with a ruler.

The three sides of a triangle determine three angles. We can only talk about the "angles of a triangle" if we think of the sides of the triangle as being rays rather than segments. The three angles of a triangle can be measured with a protractor.

The three angles of a triangle together with the three sides are called the six parts of the triangle.

EXAMPLE:



What is the sum of the measures of the 3 angles of △PQR? __________

This is △PQR

Name the 3 sides of △PQR and state their measures to the nearer tenth centimetre.

Side	Measure
$\overline{PR}$	5.8cm
____	____
____	____

Name the 3 angles of △PQR and state their measures to the nearest degree.

Angle	Measure
∠R	43°
____	____
____	____

In the example at the bottom of page 27, the sum of the measures of the three angles of $\triangle PQR$ should either equal or be very close to 180°. Your sum may differ slightly from 180° because your protractor does not allow you to make exact measurements.

> In general, the sum of the measures of the three angles determined by a triangle is 180°.

Thus, if the measures of two angles in a triangle are known, the measure of the third angle can easily be determined.

EXAMPLE: In $\triangle ABC$, $m\angle A = 35°$ and $m\angle B = 25°$. Find $m\angle C$.

Solution

$$\begin{aligned} m\angle C &= 180° - (35 + 25)° \\ &= 180° - 60° \\ &= 120° \end{aligned}$$

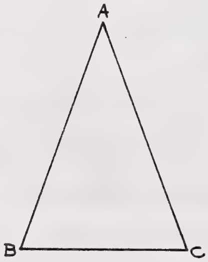
B. Classifying Triangles

Triangles can be classified in two main ways; by sides and by angles.

I. Classifying Triangles by Sides

1. An ISOSCELES TRIANGLE is a triangle with two congruent sides. The third side is called the BASE of the triangle. The angle opposite the base is called the VERTEX ANGLE and the two angles at the ends of the base are called BASE ANGLES. The base angles of an isosceles triangle are always congruent.

EXAMPLE:



$\triangle ABC$ is an isosceles triangle.

$\overline{AB} \cong \overline{AC}$ since AB = AC = _____ cm.

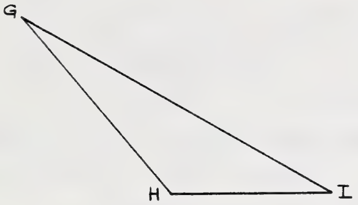
$\angle A$ is the vertex angle.

$\angle B$ and $\angle$ __ are the base angles.

$\angle B \cong \angle C$ since $m\angle B = m\angle C$ = _____°.

2. An OBTUSE TRIANGLE is a triangle with one obtuse angle.

EXAMPLE:

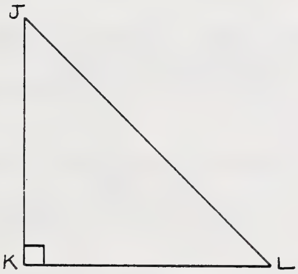


$\triangle GHI$ is an obtuse triangle.

Since $m\angle H$ = ___°, $\angle H$ is an obtuse angle.

3. A RIGHT TRIANGLE is a triangle with one right angle. In a right triangle, the side opposite the right angle is called the HYPOTENUSE and the sides of the right angle are called the LEGS.

EXAMPLE:



$\triangle JKL$ is a right triangle since

$m\angle K$ = ___°. The hypotenuse of

$\triangle JKL$ is $\overline{JL}$ and the legs are $\overline{JK}$ and

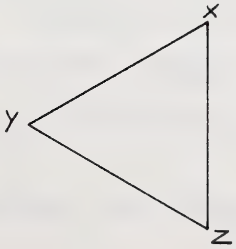
______.

4. An EQUIANGULAR TRIANGLE is a triangle with three congruent angles. An equiangular triangle is also an acute triangle.

EXAMPLE:

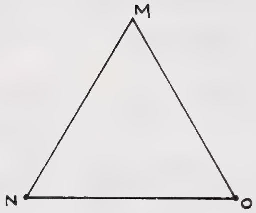
$\triangle XYZ$ is an equiangular triangle since

$m\angle X = m\angle Y = m\angle Z$ = ______°.



2. An EQUILATERAL TRIANGLE is a triangle with three congruent sides. Each of the three angles of an equilateral triangle measures 60°. An equilateral triangle is also an isosceles triangle.

EXAMPLE:



$\triangle MNO$ is an equilateral triangle.

$\overline{MN} \cong \overline{MO} \cong$ ___ since

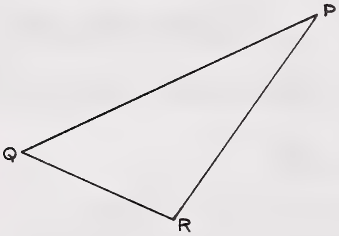
$MN = MO =$ _____ = _____ cm.

$\angle M \cong \angle N \cong \angle$ ___ since

$m\angle M = m\angle N = m\angle$ ___ = ____°.

3. A SCALENE TRIANGLE is a triangle with no congruent sides.

EXAMPLE:



$\triangle PQR$ is a scalene triangle. The three sides all have different measures.

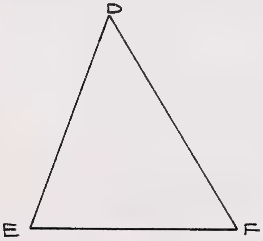
i.e. QP = ___ cm, QR = ____ cm

and PR = ___ cm.

II. Classifying Triangles by Angles

1. An ACUTE TRIANGLE is a triangle with three acute angles.

EXAMPLE:



$\triangle DEF$ is an acute triangle.

$m\angle D =$ ___°, $m\angle E =$ ___°,

$m\angle F =$ ___°.

All three angles have measures less than 90°.

Look at the triangles on pages 28 and 29 again and classify them according to their angles.

1. △ABC is a(n) ____________________ triangle.

2. △MNO is an acute triangle and an ____________________ triangle.

3. △PQR is a(n) ____________________ triangle.

Look at the triangles on pages 29 and 30 again and classify them according to their sides.

1. △DEF is a(n) ____________________ triangle.

2. △GHI is a(n) ____________________ triangle.

3. △JKL is a(n) ____________________ triangle.

4. △XYZ is an isosceles and a(n) ____________________ triangle.

Decide whether each of the following statements is true or false.

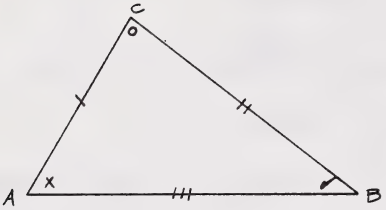
	True or False?
1. Every equilateral triangle is isosceles.	____________
2. Some obtuse triangles are isosceles.	____________
3. All right triangles are scalene.	____________
4. No scalene triangle is isosceles.	____________
5. Some right triangles are obtuse.	____________
6. No equilateral triangle is obtuse.	____________
7. All acute triangles are scalene.	____________
8. All equiangular triangles are equilateral.	____________
9. All equilateral triangles are equiangular.	____________
10. Some acute triangles are equiangular.	____________

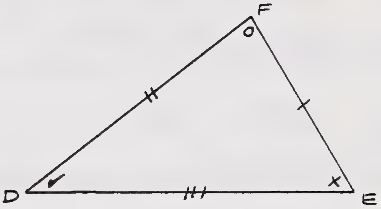
C. Congruent Triangles

> Two triangles are CONGRUENT if corresponding angles and corresponding sides are congruent.

Corresponding parts of two triangles are those parts which hold the same relative position with respect to the other parts.

Two triangles that are congruent have the same size and shape.
EXAMPLE:





These two triangles are congruent. Note that the same markings have been used on the triangles to indicate congruent parts. In this case,

$\overline{AC} \cong \overline{EF}$ $\angle C \cong \angle F$

$\overline{CB} \cong \overline{FD}$ $\angle A \cong \angle E$

$\overline{AB} \cong \overline{ED}$ $\angle B \cong \angle D$

To indicate that the two triangles are congruent, we write

$$\triangle ABC \cong \triangle EDF$$

We label the triangles in the order of the correspondence of their vertices. The above statement tells us that:

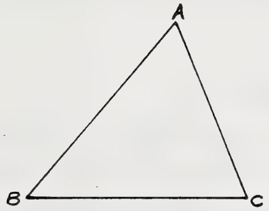
CORRESPONDS TO $A \longleftrightarrow E$, $B \longleftrightarrow D$, $C \longleftrightarrow F$

Is there a minimum number of pairs of sides or angles of two triangles that must be shown to be congruent in order to assume that the two triangles are congruent? In drawing triangles, it is unnecessary to know the measures of all six parts of the triangle before the triangle can be constructed. If we are given the measures of two sides and the included angle, two angles and the included side, or three sides, then we can construct the triangle.

I. Two Sides and the Included Angle

What do we mean by "two sides and the included angle" of a given triangle? Any two sides of a triangle determine an angle. This group of three parts is referred to as "two sides and the included angle." A given triangle has 3 groups of this type.

EXAMPLE:



In $\triangle ABC$, $\overline{AB}$, $\overline{BC}$, and $\angle B$ form a group of two sides and the included angle. Notice that the common endpoint of the two sides is the vertex of the angle. Two other groups of this type are:

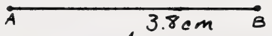
1. $\overline{BA}$, $\overline{AC}$, and $\angle A$
2. $\overline{AC}$, $\overline{BC}$, and $\angle C$

If the measures of two sides and the included angle of a triangle are known, a unique triangle is determined.

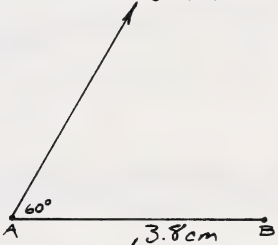
EXAMPLE: Draw $\triangle ABC$ in which $AB = 3.8$ cm, $AC = 3.2$ cm, and $m\angle A = 60°$.

Solution

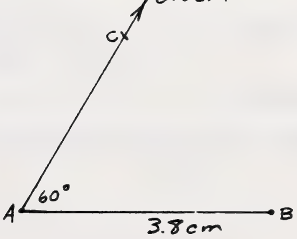
Draw $\overline{AB}$ with a measure of 3.8 cm.



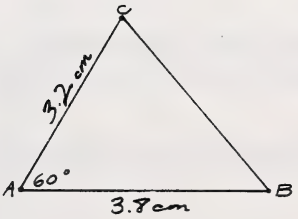
Using your protractor, construct an angle of 60° at point A.



Using your ruler, make $\overline{AC}$ 3.2 cm.



Join points C and B to form the third side of the triangle.



Since a triangle is determined if the measures of two sides and the included angle are given, this means that all triangles made with this same group of given parts must be the same size and shape. In other words, they must be congruent.

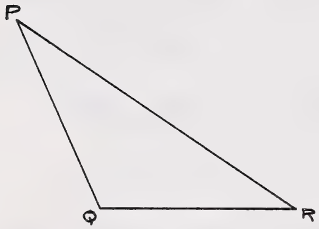
> Two triangles are congruent if two sides and the included angle of one are respectively congruent to two sides and the included angle of the other.

The above statement is often called the S.A.S. property of congruent triangles. (S.A.S. stands for "side-angle-side.")

II. Two Angles and the Included Side

What do we mean by "two angles and the included side" of a given triangle? This expression refers to a side of a triangle and the two angles at the ends of that side. A given triangle has three groups of this type.

EXAMPLE:



In $\triangle PQR$, $\angle Q$, $\overline{QR}$, and $\angle R$ form a group of two angles and the included side. Notice that the vertices of the angles are the same as the endpoints of the side.
Two other groups of this type are:

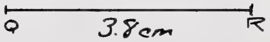
1. $\angle P$, $\overline{PQ}$, $\angle Q$
2. $\angle P$, $\overline{PR}$, $\angle R$

If the measures of two angles and the included side of a triangle are known, a unique triangle is determined.

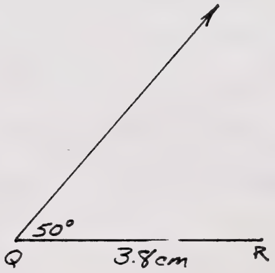
EXAMPLE: Draw $\triangle PQR$ in which $QR = 3.8$ cm, $m\angle Q = 50^\circ$, and $m\angle R = 30^\circ$.

Solution

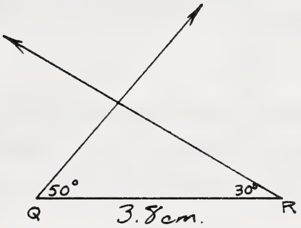
Draw $\overline{QR}$ with a measure of 3.8 cm.



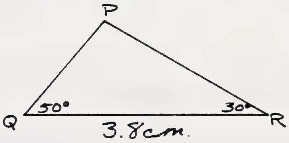
Using your protractor, construct an angle of 50° at point Q.



Using your protractor, construct an angle of 30° at point R.



Label the point of intersection of the two rays point P.



Since a triangle is determined if the measures of two angles and the included side are known, this means that all triangles made with this same group of given parts must be congruent.

> Two triangles are congruent if two angles and the included side of one are respectively congruent to two angles and the included side of the other.

The above statement is often called the A.S.A. property of congruent triangles. (A.S.A. stands for "angle-side-angle.")

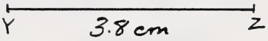
III. Three Sides

If the measures of three sides of a triangle are known, a unique triangle is determined.

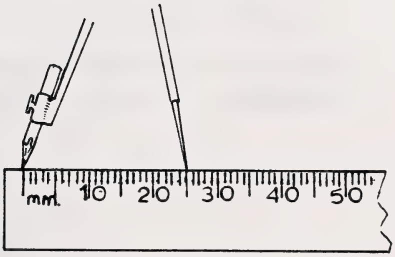
EXAMPLE: Draw $\triangle XYZ$ in which $YZ = 3.8$ cm, $XY = 2.5$ cm, $XZ = 3.2$ cm

Solution

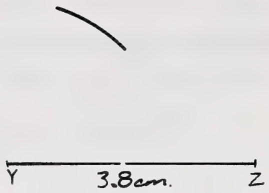
Draw $\overline{YZ}$ with a measure of 3.8 cm.



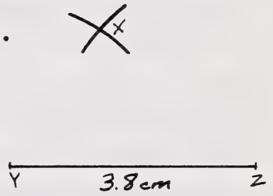
You must use a compass to copy the lengths of the other two sides. $\overline{XY}$ must measure 2.5 cm. Lay your compass along a ruler and measure off a radius of 2.5 cm.



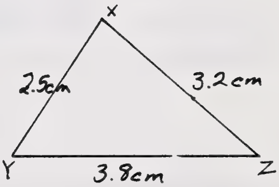
Keeping the legs of the compass 2.5 cm apart, put the metal tip of the compass at point Y and describe an arc above $\overline{YZ}$.



$\overline{XZ}$ must measure 3.2 cm. Lay your compass along a ruler and measure off a radius of 3.2 cm. Keeping the legs of the compass 3.2 cm apart, put the metal tip of the compass at point Z and describe another arc above $\overline{YZ}$. Label the point of intersection of the two arcs, X.



Join points X and Y and points X and Z to form $\triangle XYZ$.



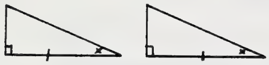
Since a triangle is determined if the measures of three sides are known, this means all triangles made with this same group of given parts must be congruent.

> Two triangles are congruent if three sides of one are congruent to three sides of the other.

The above statement is often called the S.S.S. property of congruent triangles. (S.S.S. stands for "side-side-side.")

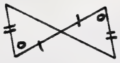
State by which property each of the following is congruent (the S.A.S. property, the A.S.A. property, or the S.S.S. property.) (Consider as equals only the parts that are so marked.)

1.

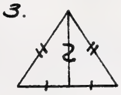


A.S.A.

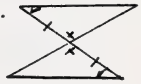
2.



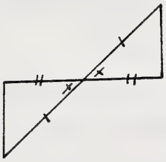
3.



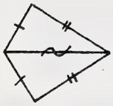
4.



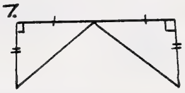
5.



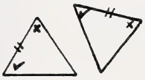
6.



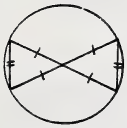
7.



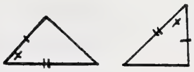
8.



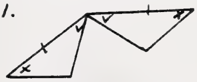
9.



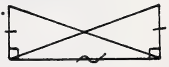
10.



11.

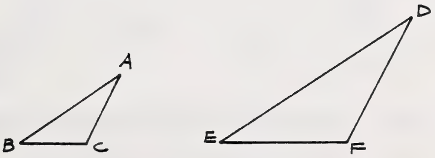


12.



D. Similar Triangles

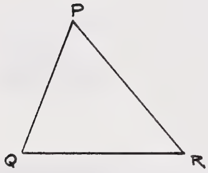
Congruent triangles have the same size and shape. However, there are many triangles that have the same shape but not necessarily the same size. Such triangles are called SIMILAR TRIANGLES. For example, $\triangle ABC$ and $\triangle DEF$ below are similar triangles since they have the same shape. $\triangle DEF$ is an enlargement of $\triangle ABC$. The symbol $\sim$ means "is similar to".

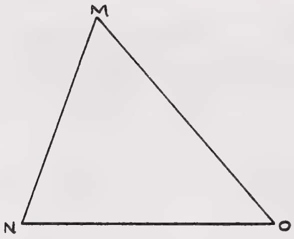


$\triangle ABC \sim \triangle DEF$

Further investigation of pairs of similar triangles reveals that for all similar triangles, corresponding angles are congruent.

EXAMPLE:





$\triangle PQR$ is similar to $\triangle MNO$. We can write:

$$\triangle PQR \sim \triangle MNO$$

$m\angle Q = m\angle N = \underline{\qquad}^{\circ}$

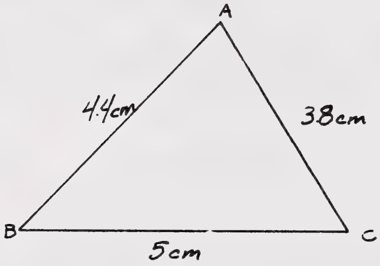
$m\angle P = m\angle M = \underline{\qquad}^{\circ}$

$m\angle R = m\angle O = \underline{\qquad}^{\circ}$

Thus, the three angles in $\triangle PQR$ are congruent to the corresponding angles in $\triangle MNO$.

An important relationship exists between the measures of corresponding sides of similar triangles. Corresponding sides are always in the same ratio.

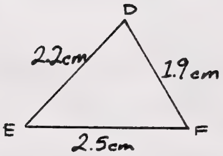
EXAMPLE:



$\triangle ABC \sim \triangle DEF$ since corresponding angles are congruent. In these triangles, note that $\overline{AB} \leftrightarrow \overline{DE}$, $\overline{AC} \leftrightarrow \overline{DF}$, and $\overline{BC} \leftrightarrow \overline{EF}$. ← CORRESPONDS TO

Let us find the ratio of the measures of correspondings sides.

$$\frac{AB}{DE} = \frac{4.4}{2.2} = \frac{\overset{2}{\cancel{4.4}}}{\underset{1}{\cancel{2.2}}} = \frac{2}{1}$$



$$\frac{AC}{DF} = \frac{3.8}{1.9} =$$

$$\frac{BC}{EF} = \frac{\qquad}{2.5}$$

In every case, the ratio of corresponding sides is $\frac{2}{1}$ (or 2:1). This means that the sides of $\triangle ABC$ are all twice as long as the corresponding sides of $\triangle DEF$.

i.e. $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = \frac{2}{1}$

In general, two triangles are SIMILAR if and only if corresponding angles are congruent and corresponding sides are in the same ratio.

A statement of equality between two ratios is called a PROPORTION. For similar triangles ABC and DEF on page 38, we can write the following proportion:

$$\left.\frac{AB}{DE} = \frac{AC}{DF}\right\}$$ This can be read, "AB is to DE as AC is to DF".

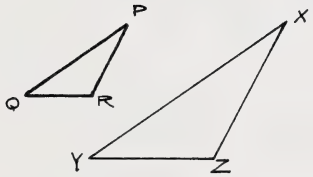
This proportion is a true statement since it compares the ratios of corresponding sides of similar triangles.

Complete the following proportions with reference to triangles ABC and DEF on page 38.

$\frac{AC}{DF} = \frac{BC}{\quad}$, $\frac{BC}{EF} = \frac{AB}{\quad}$, $\frac{DF}{AC} = \frac{EF}{\quad}$

For two similar triangles, the ratio of two sides of one triangle is equal to the ratio of corresponding sides of the other triangle.

EXAMPLE:



$\triangle PQR \sim \triangle XYZ$

$$\frac{PQ}{QR} = \frac{XY}{YZ}$$

This is a true proportion since the ratio of two sides of $\triangle PQR$ has been compared with the ratio of corresponding sides of $\triangle XYZ$.

Complete the following proportions with reference to triangles PQR and XYZ above.

$\frac{PQ}{PR} = \frac{XY}{\quad}$, $\frac{QR}{PR} = \frac{\quad}{XZ}$, $\frac{YZ}{XY} = \underline{\quad}$

When working with problems involving the theory of similar triangles, you are often faced with the task of solving proportional equations in one variable. Such equations can be solved by using the following rule of proportions.

$$\boxed{\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } a \times d = b \times c.}$$

Thus a proportional equation can be written in an alternative form by finding the cross-products of its terms and setting these cross-products equal to each other.

Note how the proportional equation $\frac{3}{5} = \frac{12}{x}$ can be solved for the variable x.

$$\frac{3}{5} = \frac{12}{x}$$

Apply rule of proportions.

$$3 \times x = 5 \times 12$$

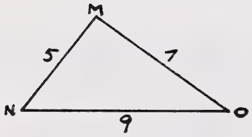
$$3x = 60$$

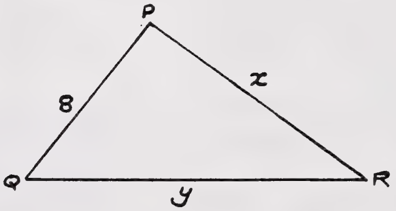
Divide both sides by 3.

$$x = 20$$

We can set up and solve proportions in order to find unknown measures of sides of similar triangles.

EXAMPLE: $\triangle MNO \sim \triangle PQR$. Find the lengths x and y.





Solution

$$\frac{MO}{MN} = \frac{PR}{PQ}$$

$$\frac{7}{5} = \frac{x}{8}$$

$$5 \times x = 7 \times 8$$

$$5x = \_\_\_\_\_\_$$

$$x = \_\_\_\_\_\_$$

$$\frac{MN}{NO} = \frac{PQ}{QR}$$

$$\frac{5}{9} = \frac{\_\_}{y}$$

$$5 \times \_\_\_ = 9 \times \_\_\_$$

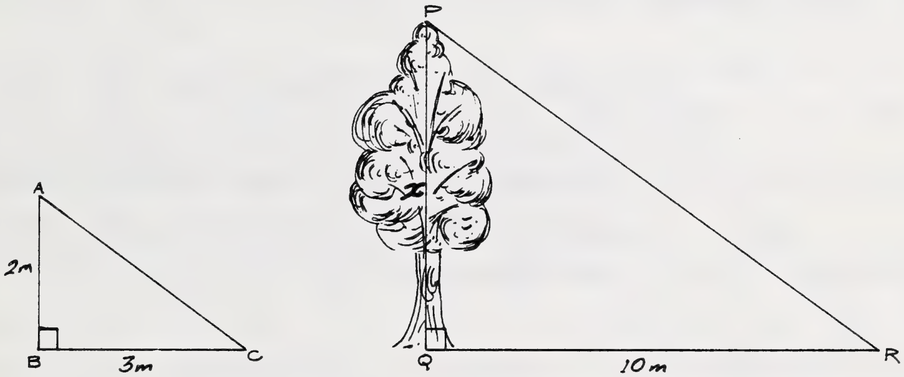
$$\_\_y = \_\_\_$$

$$y = \_\_\_\_\_\_$$

We can use what we know about similar triangles to help determine inaccessible distances.

EXAMPLE: If a vertical pole 2 m high casts a shadow 3 m long at the same time that a tree casts a shadow 10 m long, find the height of the tree.

Solution



Let the measure of the height of the tree be x m. $\triangle ABC \sim \triangle PQR$ since corresponding angles are congruent. Therefore, corresponding sides are in the same ratio.

$$\frac{AB}{PQ} = \frac{BC}{QR}$$

$$\frac{2}{x} = \frac{3}{10}$$

$$3 \times x = 2 \times 10$$

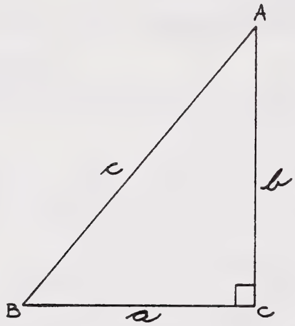
$$3x = 20$$

$$x = 6.67 \quad \text{(to 2 decimal places)}$$

The height of the tree is 6.67 m.

E. Pythagorean Theorem

Although the theorem concerning the squares on the sides of a right triangle is usually attributed to the Greek scholar Pythagoras, the practical application of this theorem was known much earlier by the Egyptian surveyors. Pythagoras was probably the first to give a formal statement of this theorem and he and his colleagues made extensive investigation into PYTHAGOREAN TRIPLETS. A Pythagorean triplet is a set of three integers so related that they represent the sides of a right-angled triangle.



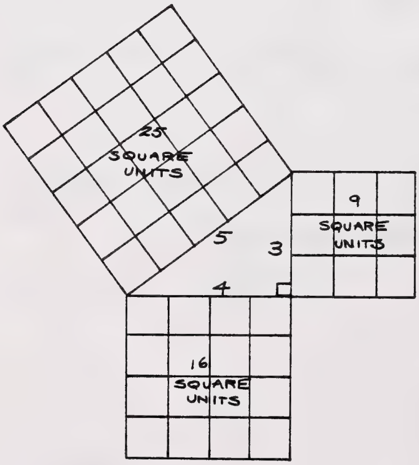
$\triangle ABC$ is a right triangle with $m\angle C = 90°$. By convention, the measure of each side is given by the small letter corresponding to the angle opposite that side. Thus, b is the side opposite $\angle B$, a is the side opposite $\angle A$, and c is the measure of the hypotenuse - the side opposite the right angle, $\angle C$.

According to the Theorem of Pythagoras,

$$c^2 = a^2 + b^2.$$

In other words, the square of the measure of the hypotenuse is equal to the sum of the squares of the measures of the other two sides.

The diagram below shows how this theorem holds for the Pythagorean triplet (3, 4, 5).



$$5^2 = 3^2 + 4^2$$

(or)

$$25 = 9 + 16$$

PYTHAGOREAN THEOREM

$$c^2 = a^2 + b^2$$

The square of the measure of the hypotenuse of a right triangle is equal to the sum of the squares of the measures of the other two sides.

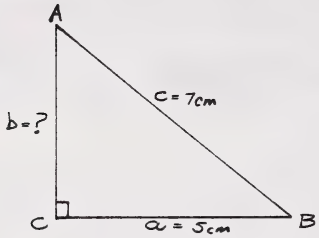
Check to see which of the following are Pythagorean triplets.

Triplet (a, b, c)	a^2	b^2	$a^2 + b^2$	c^2	Does $c^2 = a^2 + b^2$?
(6, 8, 10)	36	64	100	100	yes
(4, 5, 6)					
(12, 15, 18)					
$(1, 2, \sqrt{5})$					
$(1, \sqrt{3}, 2)$					
$(\sqrt{3}, \sqrt{4}, \sqrt{5})$					
(3, 4, 5)					
(9, 12, 15)					

If two sides of a right triangle are known, the third side may be determined by substituting in the formula $c^2 = a^2 + b^2$.

EXAMPLE: For a right triangle, the hypotenuse measures 7 cm and one leg measures 5 cm. Find the length of the other leg.

Solution



$$c^2 = a^2 + b^2$$
$$49 = 25 + b^2$$
$$b^2 = 49 - 25$$
$$b^2 = 24$$
$$b = \sqrt{24} \leftarrow \text{Only principal root is considered since lengths are positive.}$$
$$b = 4.899 \leftarrow \text{See table on page 57, lesson B.}$$

The other leg is approximately 4.899 cm.

In each question below, you are given the lengths of two sides of a right triangle. Find the length of the third side by substituting in the formula $c^2 = a^2 + b^2$. (Use table of square roots in lesson B to evaluate roots that aren't perfect squares.)

1. $a = 5$ and $b = 6$. Find c.

2. $a = 3$ and $c = 6$. Find b.

3. $c = 15$ and $b = 13$. Find a.

4. $a = 5$ and $b = 12$. Find c.

EXERCISE - Triangles

1. Fill in the blanks.

(a) A triangle can be drawn by joining three ________________ points. These points are called the ________________ of the triangle.

(b) ________________ triangles have the same size and shape while ________________ triangles have the same shape.

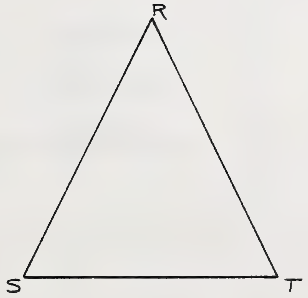
(c) According to the Pythagorean theorem, the square of the measure of the ________________ of a ________________ triangle is equal to the ____________ of the squares of the measures of the other two sides.

(d) If $\triangle ABC$ is an isosceles triangle in which $\overline{AB} \cong \overline{AC}$, then

$\angle$____ $\cong$ $\angle$____.

(e) If two angles of a triangle measure 53° and 47°, the third angle must measure ____________°.

2.



(a) Name the triangle. ________

(b) Name its sides. $\overline{RS}$, ____, ____

(c) Name its angles. ____, ____, ____

(d) Name its vertices ____, ____, ____

(e) Classify it according to its sides.

____

(f) Classify it according to its angles.

____

3. Construct the following triangles.

(a) $\triangle ABC$ in which $AB = 3$ cm, $BC = 5$ cm, and $m\angle B = 110°$.

(b) $\triangle DEF$ in which $EF = 5$ cm, $DE = 6$ cm, and $DF = 6.5$ cm.

4. In each diagram, congruent sides and angles are marked. State the property that tells you that the two triangles are congruent. Then list the three pairs of congruent angles and/or sides not used to prove the triangles congruent.

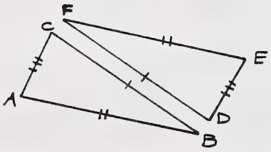
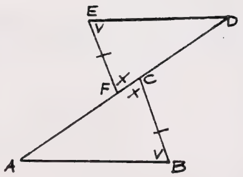
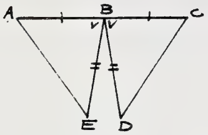
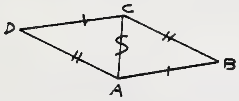
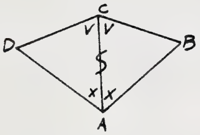
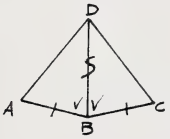
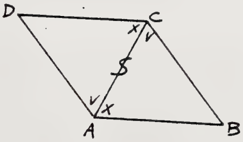
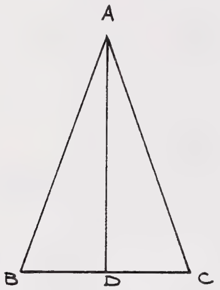
Diagram	Property	Other Congruencies
	S.S.S. property	$\angle A \cong \angle E$ $\angle C \cong \angle D$ $\angle B \cong \angle F$
	______	______ ______ ______

Diagram	Property	Other Congruencies
	____________	____________ ____________ ____________
	____________	____________ ____________ ____________
	____________	____________ ____________ ____________
	____________	____________ ____________ ____________
	____________	____________ ____________ ____________

5. In each exercise, mark the parts on the diagram that you are told are congruent. Then state the property that tells you that the two triangles are congruent.

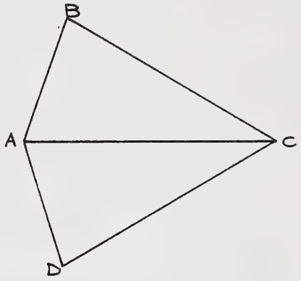
SEE PAGE 28.

(a) $\triangle ABC$ is an isosceles triangle with $\angle B \cong \angle C$. D is the midpoint of $\overline{BC}$.



$\triangle ABD \cong \triangle$_____ by the _____ property.

(b) Polygon ABCD in which diagonal AC bisects $\angle A$ and $\angle C$.

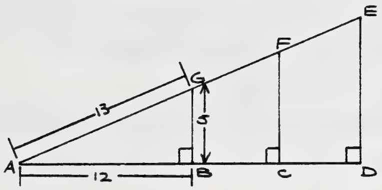


$\triangle ABC \cong \triangle$_____ by the _____ property.

6. A triangle has sides of 8 cm, 10 cm, and 15 cm. The shortest side of a similar triangle is 12 cm. How long are its other sides?

7. How high is a flagpole which casts a 15-metre shadow when a man 1.8 m tall casts a 2-metre shadow?

8.



(a) Find the ratio of CF to AC.

$$\frac{CF}{AC} = \frac{\quad}{12}$$

(b) Find the ratio of DE to AD.

$$\frac{DE}{AD} =$$

(c) Give two ratios which equal the ratio of CF to AF.

$\frac{BG}{AG}$, ______

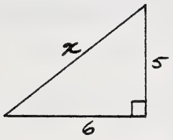
(d) Give two ratios which equal the ratio of AB to AG.

______, ______

(e) If BC = 8 cm and CD = 8.8 cm, find the length of $\overline{DE}$.

9. For each of the following right triangles, find the value of x. (Use square roots tables to evaluate irrational roots.)

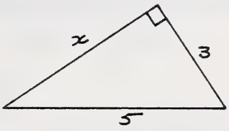
(a)



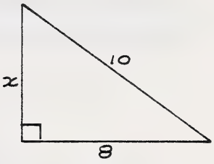
$x^2 = 5^2 + 6^2$

$x^2 =$

(b)



(c)



Topic Three: Quadrilaterals

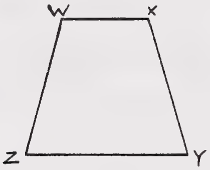
A quadrilateral is a four-sided polygon. A quadrilateral can be drawn by joining four points, no three of which lie in a straight line. Each of these points is called a vertex of the quadrilateral.

Special names are given to quadrilaterals that have pairs of sides that are equal and/or parallel.

A. Trapezoid

A TRAPEZOID is a quadrilateral with exactly one pair of opposite sides that are parallel.

Remember that parallel lines are lines in a plane that do not intersect.



This is a trapezoid WXYZ.

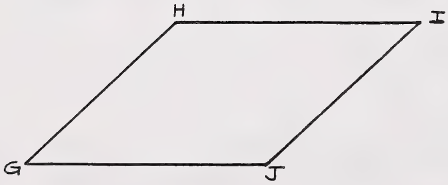
Its two sides that are parallel are

$\overline{WX}$ and ______.

B. Parallelogram

A PARALLELOGRAM is a quadrilateral with opposite sides that are parallel.

Since a parallelogram has two pairs of parallel sides, it is not a trapezoid. (A trapezoid has only one pair of parallel sides.)



This is parallelogram GHIJ. It has two pairs of parallel sides. $\overline{HG}$ is parallel to ______ and $\overline{HI}$ is parallel to ______.

Measure the 4 sides and 4 angles of this parallelogram. Note that opposite sides are congruent and opposite angles are congruent.

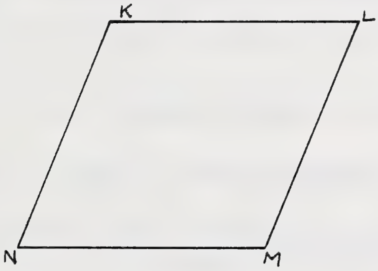
i.e. $\overline{HI} \cong$ ____ and $\overline{HG} \cong$ ____

$\angle G \cong$ ____ and $\angle H \cong$ ____

C. Rhombus

A RHOMBUS is a quadrilateral with opposite sides parallel and all sides of the same length.

A rhombus is a special kind of parallelogram. It has two pairs of parallel sides which make it a parallelogram. But, it has the added feature that all fours sides are equal in length.



This is rhombus KLMN.
It has two pairs of parallel sides.

$\overline{KL}$ is parallel to ______ and $\overline{KN}$ is parallel to ______.

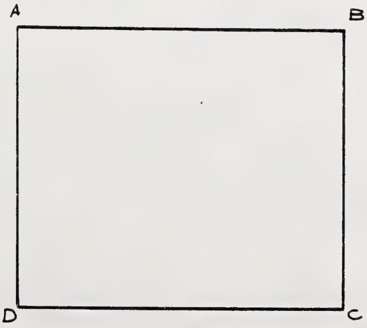
Each of the four sides has a measure of ______ cm (to one-decimal place).

Since a rhombus is a special kind of parallelogram, opposite angles of a rhombus are congruent. Thus, in rhombus KLMN, $\angle K \cong$ ___ and $\angle N \cong$ ___.

D. Rectangle

A RECTANGLE is a quadrilateral with opposite sides parallel and four right angles.

A rectangle is a special kind of parallelogram. It has two pairs of parallel sides which make it a parallelogram. But, it has the added feature that all four angles are right angles.



This is rectangle ABCD. It has two pairs of parallel sides. $\overline{AB}$ is parallel to ____ and $\overline{BC}$ is parallel to ____.

Two sides measure ____ cm and two sides measure ____ cm.

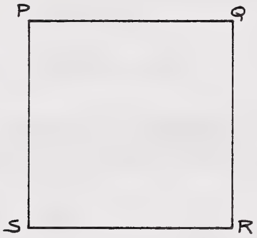
Each of the four angles has a measure of ______°.

E. Square

A square is a quadrilateral with opposite sides parallel, all sides of the same length, and four right angles.

A square is a special kind of rectangle. It has two pairs of parallel sides and four right angles which make it a rectangle. But, it has the added feature that all four sides are equal in length.

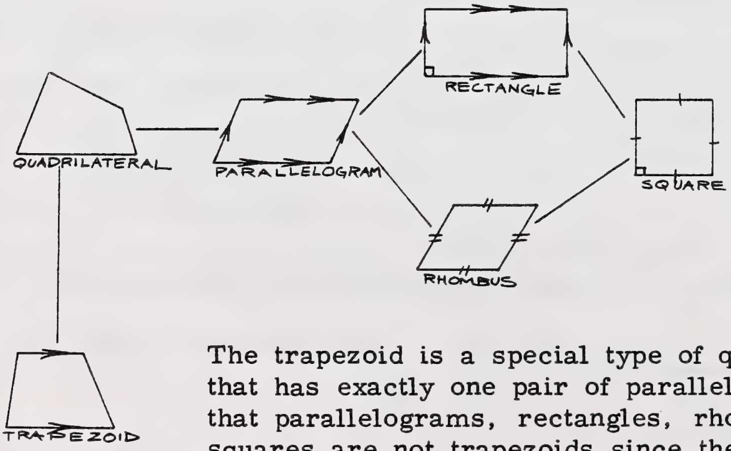
A square is also a special kind of rhombus. It has two pairs of parallel sides and four sides of the same length which make it a rhombus. But, it has the added feature that all four angles are right angles.



This is square PQRS.

It has two pairs of parallel sides. $\overline{SR}$ is parallel to ______ and $\overline{QR}$ is parallel to ______. Each of the four sides measure ______ cm. All four angles have a measure of ______°.

The following diagram may help you understand how the special kinds of quadrilaterals are related to each other. As you move to the right on the diagram, the kinds of quadrilaterals become more specialized. Any figure belongs to all the classifications that appear to the left of it on the diagram.



The trapezoid is a special type of quadrilateral that has exactly one pair of parallel sides. Note that parallelograms, rectangles, rhombuses, and squares are not trapezoids since they have two pairs of parallel sides.

EXERCISE - Quadrilaterals

1. Use the diagram at the bottom of page 52 to help you write "true" or "false" beside each of the following statements.

true (a) Every square is a parallelogram. } NOTE THAT "PARALLELOGRAM" APPEARS TO THE LEFT OF "SQUARE".

________ (b) Every parallelogram is a rectangle.

________ (c) Every rectangle is a parallelogram.

________ (d) A square is a special kind of rhombus.

________ (e) A rectangle is a special kind of square.

________ (f) Every trapezoid is a special kind of parallelogram.

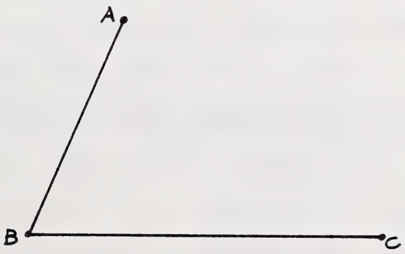
________ (g) Rectangles, squares, and rhombuses are all special kinds of parallelograms.

________ (h) Rectangles are special kinds of rhombuses.

________ (i) All quadrilaterals are parallelograms.

________ (j) Some parallelograms are squares.

2. Use your compass to locate point D in the diagram below in order to form parallelogram ABCD.
(Hint: Use the fact that opposite sides of a parallelogram are congruent. With your compass centered at A and a radius equal to BC describe an arc to the right of A. Then with your compass centered at C and a radius equal to AB, describe an arc above C that cuts the first arc at D. Join A and D and D and C to form parallelogram ABCD.)



3. Choose ONE name from the list which would best apply to the quadrilateral described.

List

trapezoid
parallelogram
rhombus
rectangle
square

(a) A quadrilateral with four right angles whose sides each measure 3.5 cm. ____________________

(b) A quadrilateral with no equal sides and only one pair of parallel sides. ____________________

(c) A quadrilateral with no right angles, opposite sides equal, and two pairs of parallel sides. ____________________

(d) A quadrilateral with four right angles, opposite sides equal, and two pairs of parallel sides. ____________________

(e) A quadrilateral with no right angles, four equal sides, and two pairs of parallel sides. ____________________

(f) A rhombus with four right angles. ____________________

(g) A parallelogram with no right angles and four equal sides.

____

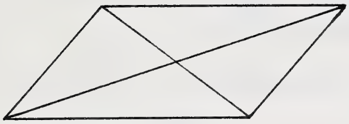
(h) A rectangle with four equal sides. ____________________

(i) A parallelogram with four right angles. ____________________

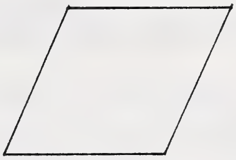
(j) A quadrilateral with opposite sides equal and parallel.

____

4. For each quadrilateral below, draw the two diagonals. (The first one has been done for you.)


Parallelogram

Rectangle


Rhombus

Square

Then answer the following questions about these diagonals. (Fill in each blank in the chart with "yes" or "no".) Measure angles and segments if you are in doubt.

	Parallelogram	Rectangle	Rhombus	Square
1. Do the diagonals bisect each other?	yes	______	______	yes
2. Are the diagonals congruent?	______	______	______	______
3. Are the diagonals perpendicular?	______	no	______	______
4. Do the diagonals bisect the angles through which they pass?	______	______	yes	______

Learn the above facts.

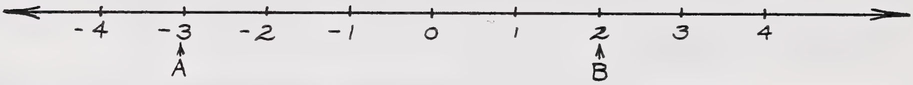
Topic Four: Coordinate Geometry

Geometry was developed many centuries before algebra. Most of the principles of Euclidean plane geometry were known to the Greeks by 300 B.C. In the seventeenth century, mathematicians applied algebraic methods to geometry linking the two together into a subject called ANALYTIC GEOMETRY or COORDINATE GEOMETRY. Then the usefulness of mathematics was greatly increased. In 1637, René Descartes, a great French mathematician, showed how the methods of algebra could be applied to geometry. In coordinate geometry, numbers are associated with points and equations with geometric figures.

Coordinate geometry is a subject that is not entirely new to you. In Lessons G and H you plotted points in the coordinate plane and learned to graph linear and quadratic equations in two variables.

A. Distance Between Two Points On the Real Number Line

You will recall that all real numbers can be represented by points on a number line.



Suppose we were interested in finding the distance between points A and B on the number line above. If we count the spaces between these two points, we find that they are 5 units apart. This distance could also be determined by subtracting the coordinates of these two points and then taking the absolute value of this difference.

$$\text{i.e.}\quad AB = \left|\overbrace{2 - (-3)}^{\leftarrow\text{DIFFERENCE OF COORDINATES}}\right| = |2 + 3| = |5| = 5$$

$$\text{(or)}\quad AB = |-3 - 2| = |-5| = 5$$

The same result is obtained regardless of the order in which the coordinates are subtracted. This is true since $|a - b| = |b - a|$ for all real values of "a" and "b".

> In general the distance between a point whose coordinate is "a" and a point whose coordinate is "b" on the real number line is:
>
> $$|a - b|$$

EXAMPLES:

1. The distance between the point representing -7 and the point representing -1 is $|(-7) - (-1)| = |-7 + 1| = |-6| = 6$.

2. The distance between the point representing 3 and the point representing -8 is $|3 - (-8)| = |3 + 8| = |11| = 11$.

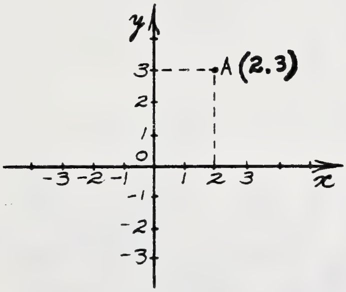
Find the distance on the real number line between pairs of points with the following coordinates.

1. -3 and - 8
   $|-3 - (-8)| =$
2. 6 and 14
3. -5 and 9
4. 2 and -7

B. Distance Between Two Points in the Coordinate Plane

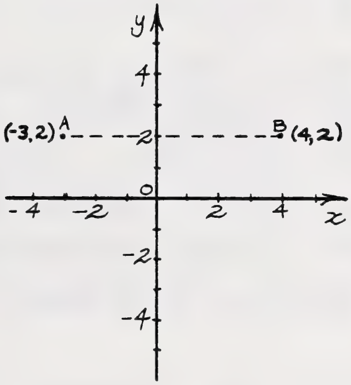
Remember that all ordered pairs of real numbers can be graphed in the coordinate plane. This plane is often called the CARTESIAN PLANE after the French mathematician, René Descartes.

EXAMPLE:



The ordered pair (2, 3) is represented by point A in the coordinate plane on the left. 2 and 3 are the COORDINATES of point A. The first coordinate, 2, is the ABSCISSA and the second coordinate, 3, is the ORDINATE.

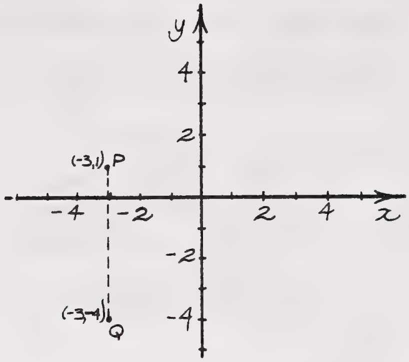
First let us consider the distance between two points in the coordinate plane that have the same ordinates (second coordinates).



The points which represent (-3, 2) and (4, 2) both have ordinates of 2. If we join these two points, we have a horizontal segment that is parallel to the x-axis. We can find the distance between these two points in the same way we found the distance between two points on the real number line. Since the ordinates are the same, we can use the absissas -3 and 4 to find the length of the segment. The distance between points A and B is equal to the absolute value of the difference of the abscissas.

$$\text{i.e. } AB = |\underbrace{-3 - 4}_{\text{DIFFERENCE OF ABSCISSAS}}| = |-7| = 7$$

Next let us consider the distance between two points in the coordinate plane that have the same abscissas (first coordinates).

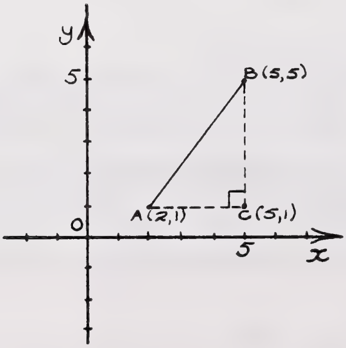


The points which represent (-3, 1) and (-3, -4) both have abscissas of -3. If we join these two points, we have a vertical segment that is parallel to the y-axis. Since the abscissas are the same, we can use the ordinates 1 and -4 to find the length of the segment. The distance between points P and Q is equal to the absolute value of the difference of the ordinates.

$$\text{i.e.} \quad PQ = |\ 1 - (-4)\ | = |\ 1 + 4\ | = |\ 5\ | = 5$$

We have learned to find the distance between two points which have either the same abscissas or the same ordinates. We will now seek a way to determine the distance between any two points in the coordinate plane.

Suppose, for example, we wished to determine the distance between the points with coordinates A(2, 1) and B(5, 5). If we join points A and B, we obtain a segment which is neither horizontal nor vertical. Thus the rules you have learned so far for determining distances cannot be applied. But, the distance can be determined if we draw a right triangle ABC with hypotenuse $\overline{AB}$.



Point C has the coordinates (5, 1) since its abscissa is the same as point B and its ordinate is the same as point A. Since $\overline{AC}$ is a horizontal segment,

$$AC = |\ 5 - 2\ | = |\ 3\ | = 3$$

Since $\overline{BC}$ is a vertical segment,

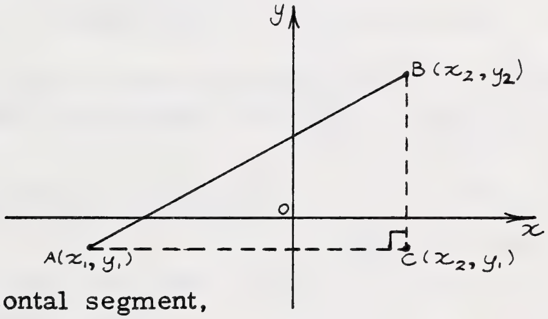
$$BC = |5 - 1| = |4| = 4$$

We can use Pythagoras' Theorem to find the length of $\overline{AB}$, the hypotenuse of the right triangle.

$$(AB)^2 = (AC)^2 + (BC)^2$$
$$(AB)^2 = 3^2 + 4^2$$
$$(AB)^2 = 9 + 16$$
$$(AB)^2 = 25$$
$$AB = 5$$

Thus, points A and B are 5 units apart.

This process can be generalized to enable us to find the distance between any two points in the coordinate plane. Consider points $A(x_1, y_1)$ and $B(x_2, y_2)$. Draw right triangle ABC with hypotenuse AB.



Since $\overline{AC}$ is a horizontal segment,

$$AC = |x_2 - x_1|.$$

Since $\overline{BC}$ is a vertical segment,

$$BC = |y_2 - y_1|.$$

According to Pythagoras' Theorem:

$$(AB)^2 = (AC)^2 + (BC)^2$$
$$(AB)^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

Absolute value signs can be dropped since all squares are positive.

$$(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$AB = \sqrt{\underbrace{(x_2 - x_1)^2}_{\text{DIFFERENCE OF ABSCISSAS}} + \underbrace{(y_2 - y_1)^2}_{\text{DIFFERENCE OF ORDINATES}}}$$

This expression is called the DISTANCE FORMULA. It tells us how to find the distance between any two points in the coordinate plane.

DISTANCE FORMULA

For two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the coordinate plane, the distance between A and B is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

MEMORIZE

EXAMPLE: Find the distance between points with coordinates (-2, 5) and (-3, -4).

Solution

$$\sqrt{\underbrace{(x_2 - x_1)^2}_{\text{DIFFERENCE OF ABSCISSAS}} + \underbrace{(y_2 - y_1)^2}_{\text{DIFFERENCE OF ORDINATES}}} = \sqrt{[\,-2 - (-3)]^2 + [\,5 - (-4)]^2}$$

$$= \sqrt{(1)^2 + (9)^2}$$

$$= \sqrt{82}$$

SEE P. 57 LESSON B.

$$= 9.055 \text{ units}$$

Self-correcting Exercise #1

Answers may be found on page 71 of this lesson.

1. Find the distance between pairs of points in the coordinate plane with the following coordinates. (Use the square root table on page 57, lesson B, to evaluate irrational roots.)

(a) (-3, 4) and (5, 9)

$$\sqrt{(-3 - 5)^2 + (4 - 9)^2}$$

$=$

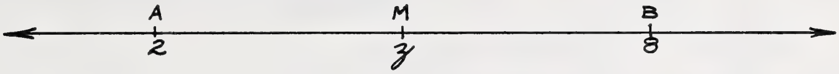
(b) (-1, -4) and (-5, -2)

(c) (3, 4) and (5, 7)

(d) (2, -8) and (-4, -7)

C. Midpoint of a Segment

First, let us consider how we would find the coordinate of the midpoint of a segment on the real number line. Consider $\overline{AB}$ with endpoints whose coordinates are 2 and 8. Suppose M is the midpoint of $\overline{AB}$ and has a coordinate of z.



Since M divides $\overline{AB}$ in half, AM = MB.
But, AM = $|z - 2| = z - 2$ (since $z > 2$) and MB = $|8 - z| = 8 - z$ (since $8 > z$).

By substitution in the equation AM = MB, we find that:

$$z - 2 = 8 - z$$
$$2z - 2 = 8$$
$$2z = 8 + 2$$
$$z = \frac{8 + 2}{2} \text{ or } 5$$

THIS IS THE AVERAGE OF 8 AND 2.

Note that the coordinate of the midpoint is in fact the average of 2 and 8, the coordinates of the endpoints of the segment.

> In general, for any two points A and B on the real number line with coordinates z_1 and z_2 respectively, the midpoint of $\overline{AB}$ has a coordinate of
>
> $$\frac{z_1 + z_2}{2}$$

EXAMPLE: What is the coordinate of the midpoint of $\overline{AB}$ whose endpoints are at -3 and 8 on the real number line?

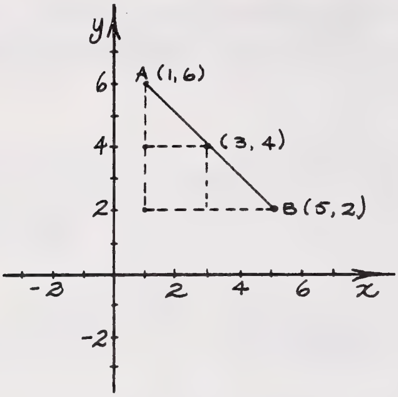
Solution

$$\frac{z_1 + z_2}{2} = \frac{-3 + 8}{2} = \frac{5}{2} = 2\frac{1}{2}$$

The midpoint occurs at the point corresponding to $2\frac{1}{2}$.

The coordinates of the midpoint of a segment in the coordinate plane can be determined in a similar manner. The abscissa of the midpoint is the average of the abscissas of the endpoints. The ordinate of the midpoint is the average of the ordinates of the endpoints.

EXAMPLE:



$$\text{Average of abscissas} = \frac{1 + 5}{2} = 3$$

$$\text{Average of ordinates} = \frac{6 + 2}{2} = 4$$

The midpoint of $\overline{AB}$ occurs at (3, 4).

In general, for any segment in the coordinate plane with endpoints (x_1, y_1) and (x_2, y_2), the midpoint of the segment occurs at

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

AVERAGE OF ABSCISSAS

AVERAGE OF ORDINATES

Self-correcting Exercise #2

Answers may be found on page 71 of this lesson.

1. Find the coordinates of the midpoints of segments with the following endpoints.

(a) (-1, 1) and (-1, 9)

$$\frac{(-1) + (-1)}{2} =$$

$$\frac{1 + 9}{2} =$$

Midpoint occurs at ________.

(b) (5, 7) and (13, 5)

Midpoint occurs at ________.

(c) (-5, 17) and (15, -2)

Midpoint occurs at ________.

(d) (-7, -5) and (1, -9)

Midpoint occurs at ________.

D. Parallel and Perpendicular Lines

In lesson G, we found that the slope of a straight-line graph can be found by taking the difference of the ordinates of two points on the line and dividing this result by the difference of the abscissas.

In general, the slope of the straight-line graph that passes through points (x_1, y_1) and (x_2, y_2) in the coordinate plane is given by:

$$m = \frac{y_2 - y_1 \leftarrow \text{DIFFERENCE OF ORDINATES}}{x_2 - x_1 \leftarrow \text{DIFFERENCE OF ABSCISSAS}}$$

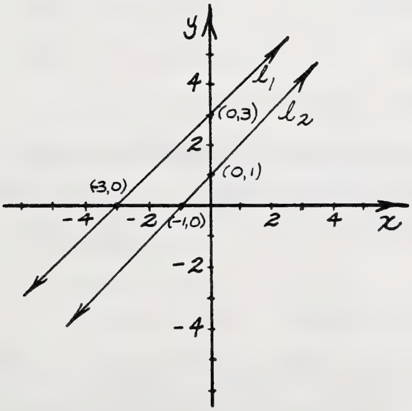
EXAMPLE: Find the slope of the straight-line graph that passes through the points $(3, -2)$ and $(-5, 7)$.

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - (-2)}{-5 - 3} = \frac{9}{-8} = -1\frac{1}{8}$$

The slope of the straight line is $-1\frac{1}{8}$.

You will recall that two straight line graphs are parallel if they have the same slope. For example, ℓ_1 and ℓ_2 below are parallel since they both have a slope of 1.



$$\text{Slope of } \ell_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 0}{0 - (-3)} = \frac{3}{3} = 1$$

$$\text{Slope of } \ell_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{0 - (-1)} = \frac{1}{1} = 1$$

We can use this information about parallel lines to solve problems involving equations of lines.

EXAMPLE: Find the equation of the line that passes through the point (-3, 4) and is parallel to the line with equation $3x - 4y + 12 = 0$.

Solution

Put the equation $3x - 4y + 12 = 0$ in the form $y = mx + b$ so that its slope m can be determined.

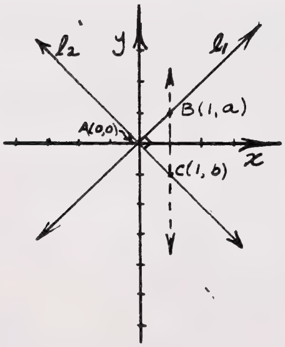
$$\begin{aligned} 3x - 4y + 12 &= 0 \\ -4y &= -3x - 12 \\ y &= \frac{3}{4}x + 3 \quad \longleftarrow m = 3/4 \end{aligned}$$

The slope of the line is $\frac{3}{4}$. Therefore, any line that is parallel to this line has a slope of $\frac{3}{4}$. The defining equation of a line with slope $\frac{3}{4}$ that passes through (-3, 4) is:

$$\begin{array}{ccccc} y - & y_1 & = & m & (x - x_1) \\ & \downarrow & & \downarrow & \quad\searrow \\ y - & 4 & = & \frac{3}{4} & [x - (-3)] \end{array}$$

$$\begin{aligned} y - 4 &= \frac{3}{4}(x + 3) \\ y - 4 &= \frac{3}{4}x + \frac{9}{4} \\ y &= \frac{3}{4}x + \frac{9}{4} + 4 \\ \underline{\underline{y}} &\underline{\underline{= \frac{3}{4}x + \frac{25}{4}}} \end{aligned}$$

Next let us consider two perpendicular lines in the coordinate plane and decide if there is any relationship between their slopes. In particular, consider two perpendicular lines ℓ_1 and ℓ_2 that intersect at the origin of the coordinate plane.



$\triangle ABC$ is a right triangle with $\angle BAC$ a right angle. The coordinates of B are (1, a) and the coordinates of C are (1, b), for some real numbers a and b. The coordinates of A are (0, 0) since point A falls at the origin.

We can use the distance formula to determine the lengths of the three sides of right $\triangle ABC$.

$$AB = \sqrt{(1 - 0)^2 + (a - 0)^2} = \sqrt{(1)^2 + a^2} = \sqrt{1 + a^2}$$

$$BC = \sqrt{(1 - 1)^2 + (a - b)^2} = \sqrt{(a - b)^2} = |a - b|$$

$$AC = \sqrt{(1 - 0)^2 + (b - 0)^2} = \sqrt{(1)^2 + b^2} = \sqrt{1 + b^2}$$

According to Pythagoras' Theorem,

$$\begin{aligned}
(BC)^2 &= (AB)^2 + (AC)^2 \\
|a - b|^2 &= \left(\sqrt{1 + a^2}\right)^2 + \left(\sqrt{1 + b^2}\right)^2 \\
(a - b)^2 &= 1 + a^2 + 1 + b^2 \\
a^2 - 2ab + b^2 &= 2 + a^2 + b^2 \\
-2ab &= 2 \\
ab &= -1
\end{aligned}$$

Note that "ab" refers to the product of the slopes of the two lines since:

$$\text{slope of } \ell_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{a - 0}{1 - 0} = \frac{a}{1} = a$$

$$\text{slope of } \ell_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{b - 0}{1 - 0} = \frac{b}{1} = b$$

Thus, $ab = -1$ tells us that the product of the slopes of two perpendicular lines is -1.

Also, since $ab = -1$, we know that $b = \frac{-1}{a}$. This tells us that if the slope of a given line is a, the slope of a line that is perpendicular to it is $\frac{-1}{a}$. In other words, the slopes of two lines that are perpendicular to each other are negative reciprocals of each other. For example, if the slope of a given line is $\frac{1}{3}$, the slope of a line that is perpendicular to it is -3. (Note that $\frac{1}{3} \times -3 = -1$.)

SLOPES OF PARALLEL OR PERPENDICULAR LINES

Two parallel lines have equal slopes. Two perpendicular lines have slopes that are negative reciprocals of each other.

We can use this information about perpendicular lines to solve problems involving equations of lines.

EXAMPLE: Find the equation of the line on A(-3, 2) which is perpendicular to the line on A and B(1, 5).

Solution

Slope of line on A and B $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{1-(-3)} = \frac{3}{4}$

The slope of a line perpendicular to the line on A and B is $\frac{-4}{3}$.

The equation of a line on A(-3, 2) with a slope of $\frac{-4}{3}$ is:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= \frac{-4}{3}(x + 3) \\ y - 2 &= \frac{-4}{3}x - 4 \\ y &= \frac{-4}{3}x - 2 \end{aligned}$$

Self-correcting Exercise #3

Answers may be found on page 72 of this lesson.

1. State the slopes of lines parallel to lines with slopes of

(a) 3 _____ (b) -5 _____ (c) $\frac{1}{2}$ _____ (d) $\frac{-3}{4}$ _____

2. State the slopes of lines perpendicular to lines with slopes of

(a) 3 _____ (b) -5 _____ (c) $\frac{1}{2}$ _____ (d) $\frac{-3}{4}$ _____

3. Without using the Distance Formula, prove that quadrilateral ABCD is a parallelogram if its coordinates are A(-5, -2), B(1, -1), C(4, 4), and D(-2, 3).

Slope of $\overline{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-(-1)}{-5-1} =$

Slope of $\overline{DC} = \frac{y_2 - y_1}{x_2 - x_1} =$

Is the slope of $\overline{AB}$ equal to the slope of $\overline{DC}$? _____

Slope of $\overline{BC} =$

Slope of $\overline{AD} =$

Is the slope of $\overline{BC}$ equal to the slope of $\overline{AD}$? _____

Therefore, ABCD is a parallelogram since opposite sides are parallel.

4. Without using the Distance Formula, prove that $\triangle ABC$ is a right triangle with right angle at A if its coordinates are A(-2, 5), B(6, 8) and C(1, -3).

Slope of $\overline{AB} = \frac{5 - 8}{-2 - 6} =$

Slope of $\overline{AC} =$

Are these slopes negative reciprocals of each other? ________
(i.e. is the product of the slopes -1?)

Since $\overline{AB}$ is perpendicular to _____, $\angle A$ is a right angle and

$\triangle ABC$ is a _______ triangle.

EXERCISE - Coordinate Geometry

1. In each exercise, you are given the slopes of two lines. Tell whether the lines are parallel, perpendicular, or neither parallel nor perpendicular.

(a) $m_1 = \frac{2}{3}$; $m_2 = \frac{3}{2}$

____

(b) $m_1 = \frac{5}{7}$; $m_2 = \frac{10}{14}$

____

(c) $m_1 = \frac{3}{2}$; $m_2 = \frac{-3}{2}$

____

(d) $m_1 = -1$; $m_2 = 1$

____

(e) $m_1 = \frac{-9}{2}$; $m_2 = \frac{2}{9}$

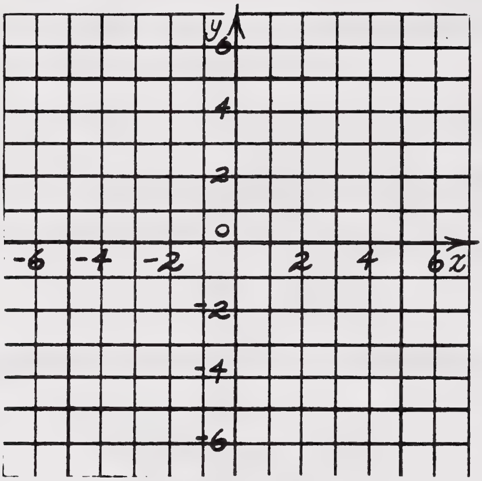
____

(f) $m_1 = 3\frac{2}{8}$; $m_2 = 3.25$

____

2. Show that the lines represented by the equations $3x + 7y - 21 = 0$ and $14x - 6y + 28 = 0$ are perpendicular to each other.

3. In the coordinate plane below, graph the triangle having vertices at A(-4, 5), B(3, 2) and C(-1, 1).



(a) Use the distance formula to find the lengths of the three sides of the triangle. (Use square root table to evaluate irrational roots.)

AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3 + 4)^2 + (2 - 5)^2} =$

AC =

BC =

(b) Determine the coordinates of the midpoints of each side.

Midpoint of $\overline{AB}$ = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4 + 3}{2}, \frac{5 + 2}{2}\right) =$

Midpoint of $\overline{AC}$ =

Midpoint of $\overline{BC}$ =

(c) Determine the slope of each side.

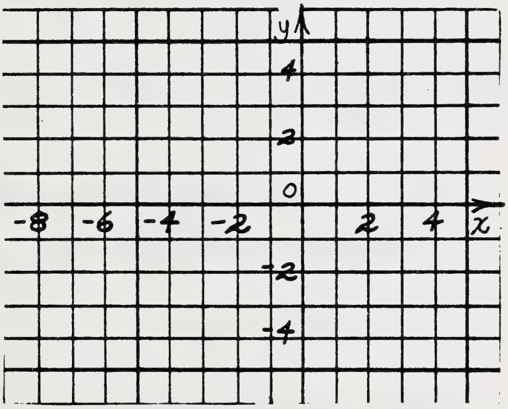
Slope of $\overline{AB}$ = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 5}{3 + 4} =$

Slope of $\overline{AC}$ =

Slope of $\overline{BC}$ =

4. Find the equation of the line on A(2, -3) and parallel to the line on B(-4, 0) and C(2, 5).

5. In the coordinate plane provided, graph the triangle having vertices at A(-7, 4), B(3, -2), and C(1, 2). Then use the midpoint formula and distance formula to find the length of the line segment from the vertex C to M, the midpoint of $\overline{AB}$.



6. Find the equation of the line through (6, -2) that is perpendicular to a line defined by $5x - 2y + 12 = 0$.

7. Triangle ABC has vertices whose coordinates are A(2, 2), B(8, 4), and C(3, 9). Show that:

(a) AC = BC.

$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} =$

$BC =$

(b) the coordinates of the midpoint M of $\overline{AB}$ are (5, 3).

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) =$

(c) $\overline{CM} \perp \overline{AB}$.

"IS PERPENDICULAR TO"

Slope of $\overline{CM}$ =

Slope of $\overline{AB}$ =

Key to Self-correcting Exercises in Lesson J

Exercise #1, page 60

1. (a) $\sqrt{(-3 - 5)^2 + (4 - 9)^2}$

$= \sqrt{(-8)^2 + (-5)^2}$

$= \sqrt{64 + 25}$

$= \sqrt{89}$

$\doteq \underline{9.434}$

(b) $\sqrt{[-1 - (-5)]^2 + [-4 - (-2)]^2}$

$= \sqrt{(-1 + 5)^2 + (-4 + 2)^2}$

$= \sqrt{(4)^2 + (-2)^2}$

$= \sqrt{16 + 4}$

$= \sqrt{20}$

$\doteq \underline{4.472}$

(c) $\sqrt{(3 - 5)^2 + (4 - 7)^2}$

$= \sqrt{(-2)^2 + (-3)^2}$

$= \sqrt{4 + 9}$

$= \sqrt{13}$

$\doteq \underline{3.606}$

(d) $\sqrt{[2 - (-4)]^2 + [-8 - (-7)]^2}$

$= \sqrt{(2 + 4)^2 + (-8 + 7)^2}$

$= \sqrt{(6)^2 + (-1)^2}$

$= \sqrt{36 + 1}$

$= \sqrt{37}$

$\doteq \underline{6.083}$

Exercise #2, page 62

1. (a) $\frac{(-1) + (-1)}{2} = \frac{-2}{2} = -1$

$\frac{1 + 9}{2} = \frac{10}{2} = 5$

Midpoint occurs at $\underline{(-1, 5)}$.

(b) $\frac{5 + 13}{2} = \frac{18}{2} = 9$

$\frac{7 + 5}{2} = \frac{12}{2} = 6$

Midpoint occurs at $\underline{(9, 6)}$.

(c) $\frac{-5 + 15}{2} = \frac{10}{2} = 5$

$\frac{17 + (-2)}{2} = \frac{15}{2} = 7\frac{1}{2}$

Midpoint occurs at $\underline{(5, 7\frac{1}{2})}$.

(d) $\frac{-7 + 1}{2} = \frac{-6}{2} = -3$

$\frac{-5 + (-9)}{2} = \frac{-14}{2} = -7$

Midpoint occurs at $\underline{(-3, -7)}$.

Exercise #3, page 66

1. (a) 3 (b) -5 (c) $\frac{1}{2}$ (d) $\frac{-3}{4}$

(Remember that if two lines are parallel, their slopes are the same.)

2. (a) $\frac{-1}{3}$ (b) $\frac{1}{5}$ (c) -2 (d) $\frac{4}{3}$

(Remember that two lines are perpendicular if their slopes are negative reciprocals of each other. In other words, the product of their slopes is -1.)

3. Slope of $\overline{AB} = \frac{-2-(-1)}{-5-1} = \frac{-2+1}{-6} = \frac{-1}{-6} = \frac{1}{6}$

Slope of $\overline{DC} = \frac{4-3}{4-(-2)} = \frac{1}{4+2} = \frac{1}{6}$

Slope of $\overline{AB}$ = Slope of $\overline{DC}$

Slope of $\overline{BC} = \frac{-1-4}{1-4} = \frac{-5}{-3} = \frac{5}{3}$

Slope of $\overline{AD} = \frac{-2-3}{-5-(-2)} = \frac{-5}{-5+2} = \frac{-5}{-3} = \frac{5}{3}$

Slope of $\overline{BC}$ = Slope of $\overline{AD}$

Therefore, ABCD is a parallelogram since opposite sides are parallel.

4. Slope of $\overline{AB} = \frac{5-8}{-2-6} = \frac{-3}{-8} = \frac{3}{8}$

Slope of $\overline{AC} = \frac{5-(-3)}{-2-1} = \frac{5+3}{-3} = \frac{8}{-3} = \frac{-8}{3}$

The slopes are negative reciprocals (whose product is -1).

Since $\overline{AB}$ is perpendicular to $\overline{AC}$, $\angle A$ is a right angle and $\triangle ABC$ is a right triangle.

End of Lesson J

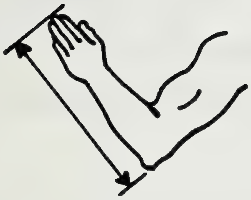
Lesson K

LESSON K: THE METRIC SYSTEM; TRIGONOMETRY; STATISTICS

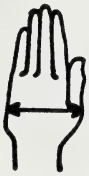
TOPIC ONE: THE METRIC SYSTEM

A. Introduction

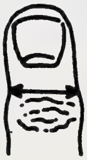
In Canada the metric system of measurement, based on metric units has only recently been adopted. It replaced the Imperial System which was based on units such as inches, pounds and gallons. The Imperial System had developed over a long period of time from convenient things such as various parts of the body. Some of these measurement standards are shown below.



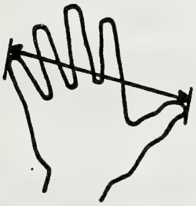
cubit



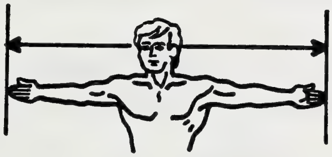
hand



digit



span



fathom

As nations developed, each chose its own system and units of measurement. Within various trades and occupations specific measuring systems were also developed. For example, surveyors used measures called rods and chains; apothecaries used minims and drams; while mariners used fathoms, knots and cable lengths. England gradually refined its system of weights and measures, adopting the most suitable units and abandoning others. The resulting Imperial System of measurement proved adequate for the needs of trade for a great number of years. However, many of the units are complicated and poorly related.

After the French Revolution of 1789, France discarded its old measures and adopted a new system of measurement, based on the metre. The **METRE** was defined to be **one ten-millionth of the measure of the meridian of longitude which runs through Paris, extending from the North Pole to the equator.** Later certain errors in the metre were found since the earth isn't quite round. The metre was then redefined in terms of the wavelength of light emitted by an atom of krypton - 86. This wavelength became the basis for an international system of measurement called SI **(Le Système International d'Unités.)** The worldwide adoption of SI would give a universal measurement language, understood in all nations.

The metric system is simple and logical since all measures are related to each other, and larger and smaller units in each series of measures are related by powers of 10. Consequently, in converting from one metric unit to another, all that is necessary is a movement of the decimal point.

B. Metric Units

The **METRE** (m) is the basic unit of length.

The **LITRE** (L) is the basic unit of capacity and is defined as **the capacity of a cube measuring one decimetre (one tenth of a metre) on a side.**

The **KILOGRAM** (kg) is the most commonly used unit of weight or mass, and is defined as **the mass of water in a litre container.**

The metric system uses a series of prefixes to relate larger or smaller units to the basic unit. The chart on page 3 lists the SI units for length, capacity and mass. **MEMORIZE** the symbol used to represent each metric unit as well as the word name.

Some Metric Conventions to Know

1. A period is never used after a metric symbol, except when the symbol occurs at the end of a sentence. (e.g. km, cm, g, kg)
2. Never add **"s"** to pluralize metric symbols.
3. The symbol for litre is a capital **L.**
4. Use spaces, not commas, to separate numbers with more than four digits into blocks of 3 digits with respect to the decimal point.

   e.g. 15 579.382 04

The starred units (*) in the chart below indicate the most commonly used units.

METRIC UNITS IN WORDS AND SYMBOLS

Units of Length		Units of Capacity		Units of Mass	
***kilometre**	**km**	***kilolitre**	**kL**	***kilogram**	**kg**
hectometre	hm	hectolitre	hL	hectogram	hg
decametre	dam	decalitre	daL	decagram	dag
***metre**	**m**	***litre**	**L**	***gram**	**g**
decimetre	dm	decilitre	dL	decigram	dg
***centimetre**	**cm**	centilitre	cL	centigram	cg
***millimetre**	**mm**	***millilitre**	**mL**	***milligram**	**mg**

Memorize this chart!

Practice writing it out and get someone to drill you on it. (For example, do you know what unit is 2 steps below "metre" in the chart?)

In the following discussion, fill in all blanks.

In the chart above, the units become **larger** as you move **upward.** Each unit is **ten times** larger than the unit immediately below it. Ⓐ

Ⓐ Since each unit is ten times larger than the unit immediately below it, we must divide by 10 to move up the chart one step.

To move down one step, we multiply by 10.

ie,

250 mm = (250 ÷ 10) cm
= 25 cm
1350 cm = (1350 × 10) mm
= 13 500 mm

EXAMPLES:

1 km = 10 hm **1 g = 10 dg**
1 cm = 10 mm **1 cg = 10 mg**

(1) 1 dm = 10 _____
(2) 1 kg = _____ hg
(3) 1 _____ = 10 mL
(4) 1 _____ = 10 hm
(5) 1 _____ = 10 g
(6) 1 cm = 10 _____

In the chart above, each unit is **100 times** larger than the unit **two steps** below it and **1000 times** larger than the unit **three steps** below it, and so on.

EXAMPLES:

Ⓑ**1 km = 100 dam**
1 L = 100 cL

Ⓒ**1 km = 1000 m**
1 daL = 1000 cL

Ⓑ The smaller unit is 2 steps below.

Ⓒ The smaller unit is 3 steps below.

(7) 1 m = _____ mm
(8) 1 _____ = 100 dL
(9) 1 kg = _____ dag
(10) 1 g = 100 _____
(11) 1 L = 1000 _____
(12) 1 _____ = 1000 mg

In the metric unit chart on page 3, the units became **smaller** as you moved **downward.** Each unit is **one-tenth** (0.1) as large as the unit immediately above it, **one hundredth** (0.01) as large as the unit **two steps** above it, **one-thousandth** (0.001) as large as the unit **three steps** above it, and so on.

EXAMPLES:

1 mm = 0.1 cm **1 L = 0.001 kL**
1 dm = 0.01 dam **1 g = 0.001 kg**

(13) 1 cm = 0.1 _____
(14) 1 _____ = 0.1 m
(15) 1 mg = _____ g
(16) 1 mL = _____ cL
(17) 1 cg = _____ g
(18) 1 cm = 0.01 _____

Give three **common** SI units used to measure the following.

(19) length: millimetre, _____________, _____________
(20) capacity: _____________, _____________, _____________
(21) mass: _____________, _____________, _____________

Check your answers with those given on page 74.

C. Metric Prefixes

The table below lists the metric prefixes with corresponding symbols and meanings. **MEMORIZE** these prefixes and their meanings so that you can make correct metric conversions quickly.

Prefix	**Symbol**	**Meaning in Words**	**Meaning in Numbers**	**Power of 10**
mega	**M**	million	1 000 000	10^6
kilo	**k**	thousand		
hecto	**h**	hundred		
deca	**da**	ten		
deci	**d**	tenth		10^{-1}
centi	**c**	hundredth		10^{-2}
milli	**m**	thousandth		10^{-3}
micro	μ	millionth	0.000 001	10^{-6}

Self-correcting Exercise #1

1. In the chart on page 4, fill in the blanks.

2. Each metric prefix is attached to a basic metric unit. Complete the following statements.

(a) The prefix "kilo" means **thousand.** Attach the prefix "kilo" to the metric unit **metre** to make the word

"kilometre," which means __________ m.

"Kilogram" means __________ g.

"Kilolitre" means __________ L.

(b) The prefix "centi" means ______________ or 0.01.

Thus "centimetre" means __________ m.

"Centigram" means __________ g.

"Centilitre" means __________ L.

(c) The prefix "milli" means ______________ or _________.

Thus "millimetre" means __________ m.

"Milligram" means __________ g.

"Millilitre" means __________ L.

3. Fill in the blanks.

Word Name	**SI Symbol**
millimetre	__________
__________	cm
__________	dm
metre	__________
decametre	__________
__________	hm
kilometre	__________
kilogram	__________
__________	g
milligram	__________
__________	kL
litre	__________
__________	mL
centilitre	__________

4. **Fill in the blanks.**

(a) The metric **prefixes** that indicate multiples of the basic unit that are less than one are ____________, ____________, ____________ and ____________.

(b) The metric **prefixes** that indicate multiples of the basic unit that are greater than one are ____________, ____________, ____________ and ____________.

5. **Circle the SMALLER unit in each pair of units. ("a" part is done for you.)**

(a) hectometre, [✓ decimetre]

(b) millilitre, hectolitre

(c) millimetre, metre

(d) gram, decigram

(e) kilogram, decagram

(f) millimetre, centimetre

(g) centimetre, decametre

(h) decagram, decigram

(i) litre, kilolitre

(j) litre, hectolitre

Check your answers with those on pages 74 and 75.

D. The Metre and Smaller Units of Length

The **METRE** is the basic unit of length. A metre is about half the height of a door. Units of length smaller than a metre are the decimetre, the centimetre and the millimetre.

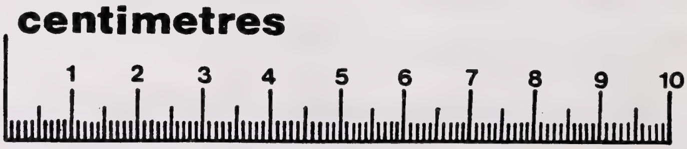
A **DECIMETRE** is one-tenth of a metre (0.1 m).

A **CENTIMETRE** is one-hundredth of a metre (0.01 m).

A **MILLIMETRE** is one-thousandth of a metre (0.001 m). 1 mm is about the thickness of a dime.

The ruler shown below is divided into ten centimetres. Each centimetre is divided into 10 parts called millimetres. Thus, each millimetre is equal to one tenth of a centimetre or 0.1 cm.

Since ten centimetres are equal to one decimetre, the entire ruler length shown is equivalent to one decimetre.



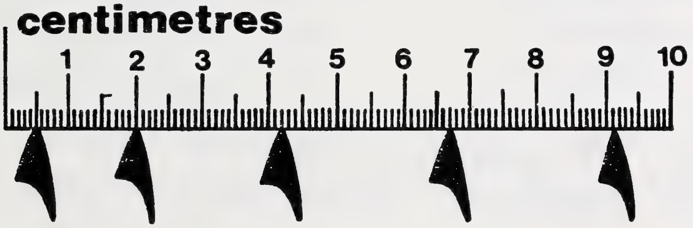
Self-correcting Exercise #2

1. **Fill in the blanks.**

(a) In the ruler diagram on page 6, the number of centimetres shown is _______ cm and the number of decimetres shown is _____ dm.

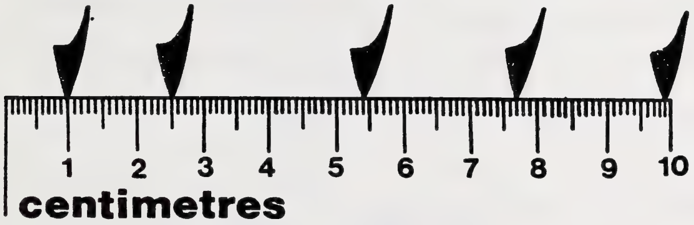
(b) The number of millimetres in one centimetre is _______ mm and the total number of millimetres shown on the ruler diagram is _______ mm.

(c) Give the ruler measure in **centimetres** at the points marked.



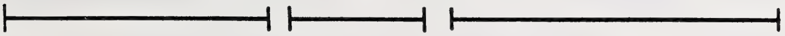
0.5 cm, _____ cm, _____ cm, _____ cm, _____ cm

(d) Give the ruler measure in **millimetres** at the points marked.



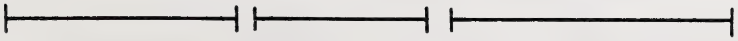
10 mm, _____ mm, _____ mm, _____ mm, _____ mm

(e) Measure each segment to the nearer **centimetre**.



____ cm, ____ cm, ____ cm

(f) Measure each segment to the nearer **millimetre**.



____ mm, ____ mm, ____ mm

2. **Convert the following measures from metres to decimetres, from metres to centimetres or from metres to millimetres.** Remember that in multiplying by 10 you move the decimal point 1 place to the **right**, by 100 two places to the right and by 1000 three places to the right.

1 m = 100 cm

9 m = 900 cm

3.6 m = _____ cm

0.4 m = _____ cm

1 m = 10 dm

5 m = 50 dm

2.2 m = _____ dm

0.7 m = _____ dm

1 m = 1000 mm

3 m = 3000 mm

2.7 m = _____ mm

4.75 m = _____ mm

3. **Convert the following measures from decimetres to metres, from centimetres to metres, or from millimetres to metres.** Remember that in multiplying by 0.1 you move the decimal point 1 place to the **left,** by 0.01 two places to the left, by 0.001 three places to the left.

1 cm = 0.01 m

50 cm = 0.5 m

200 cm = _____ m

1490 cm = _____ m

1 dm = 0.1 m

25 dm = 2.5 m

5.7 dm = _____ m

378 dm = _____ m

1 mm = 0.001 m

4 500 mm = 4.5 m

910 mm = _____ m

75 mm = _____ m

Check your answers with those on page 75.

E. The Metre and Larger Units of Length

The **METRE** is the basic unit of length. The units of length **larger** than a metre are the decametre, the hectometre and the kilometre. The kilometre, which is the most commonly used of these three units, is about the length of a brisk ten-minute walk.

A **DECAMETRE** is ten metres (10 m).

A **HECTOMETRE** is one hundred metres (100 m).

A **KILOMETRE** is one thousand metres (1000 m).

1 dam = 10 m   **1 hm = 100 m**   **1 km = 1000 m**

Self-correcting Exercise #3

Fill in the blanks.

(1) The prefix "kilo" means ______________, so to convert kilometres to metres you must ______________ by the number __________.

(2) 7 km = ______________ m; 15 km = ______________ m;
0.5 km = ______________ m

(3) The prefix "hecto" means ______________,so to convert hectometres to metres you must ______________ by the number __________.

(4) 9 hm = ______________ m; 25 hm = ______________ m;
0.7 hm = ______________ m.

(5) The prefix "deca" means ______________, so to convert decametres to metres you must ______________ by the number __________.

(6) 3 dam = ______________ m;10.8 dam = ______________ m;
64 dam = ______________ m.

(7) A metre is one-tenth of a decametre.

1 m = 0.1 dam

To convert metres to decametres you must multiply by the decimal number ______________.

(8) 19 m = ______________ dam; 345 m = ______________ dam;
6 m = ______________ dam.

(9) A metre is one one-hundredth of a hectometre.

$$\boxed{1\ \text{m} = 0.01\ \text{hm}}$$

To convert metres to hectometres you must __________ by the decimal number __________.

(10) 136 m = ______________ hm; 69 m = ______________ hm;
4 m = ______________ hm.

(11) A metre is one one-thousandth of a kilometre.

$$\boxed{1\ \text{m} = 0.001\ \text{km}}$$

To convert metres to kilometres you must __________ by the decimal number __________.

(12) 8000 m = ______________ km; 270 m = ______________ km;
85 m = ______________ km.

Check your answers with those on page 76.

Exercise - The Metric System

1. **Fill in the blanks.**

(a) 1 _____ = 1000 g
(b) 1 daL = 10 _____
(c) 1 L = 0.001 _____
(d) 10 000 mm = _____ m
(e) 1 m = _____ cm
(f) 1 _____ = 0.1 L
(g) 1 _____ = 10 dL
(h) 5 mm = _____ m
(i) 100 m = _____ cm
(j) 8 cm = _____ m
(k) 1 dm = _____ m
(l) 25.5 m = _____ cm
(m) 16 m = _____ dm
(n) 1500 dm = _____ m
(o) 3000 cm = _____ m
(p) 19 m = _____ mm
(q) 0.8 m = _____ mm
(r) 67 m = _____ dm

2. **Complete the following.**

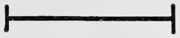
(a) The SI symbol kg represents ____________________.
(b) A millilitre is represented in SI by ______________.
(c) The SI symbol cm represents ____________________.
(d) A kilolitre is represented in SI by ______________.
(e) The SI symbol for litre is ____________________.

3. Circle the decimal number which makes each sentence true.

(a) 1 cm is (0.01, 100) m
(b) 1 m is (0.01, 100) cm
(c) 1 mm is (0.001, 1000) m
(d) 1 m is (0.001, 1000) mm
(e) 1 mm is (0.1, 10) cm
(f) 1 cm is (0.1, 10) mm
(g) 1 kL is (0.001, 1000) L
(h) 1 m is (0.001, 1000) km
(i) 1 L is (0.001, 1000) mL
(j) 1 L is (0.001, 1000) kL
(k) 1 g is (0.001, 1000) kg
(l) 1 mL is (0.001, 1000) L
(m) 1 mg is (0.001, 1000) g
(n) 1 kg is (0.001, 1000) g
(o) 1 km is (0.001, 1000) m
(p) 1 g is (0.001, 1000) mg

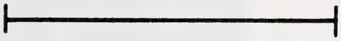
4. Measure the following segments as specified.

(a)



______ mm

(b)



______ mm

5. Draw segments to the lengths specified using a ruler.

(a) 51 mm

(b) 3.8 cm

F. Metric Units of Area

So far we have been discussing metric units of length (linear units) which involve only **one** dimension. Now we will consider the metric units used in measuring area. Area is found by multiplying the **two** dimensions, length and width, and is always measured in **SQUARE UNITS**. Some metric units of area are: **square centimetres, square metres,** and **square kilometres.**

SI symbols for area emphasize the 2-dimensional aspect by showing the "2" as an exponent.

EXAMPLES:

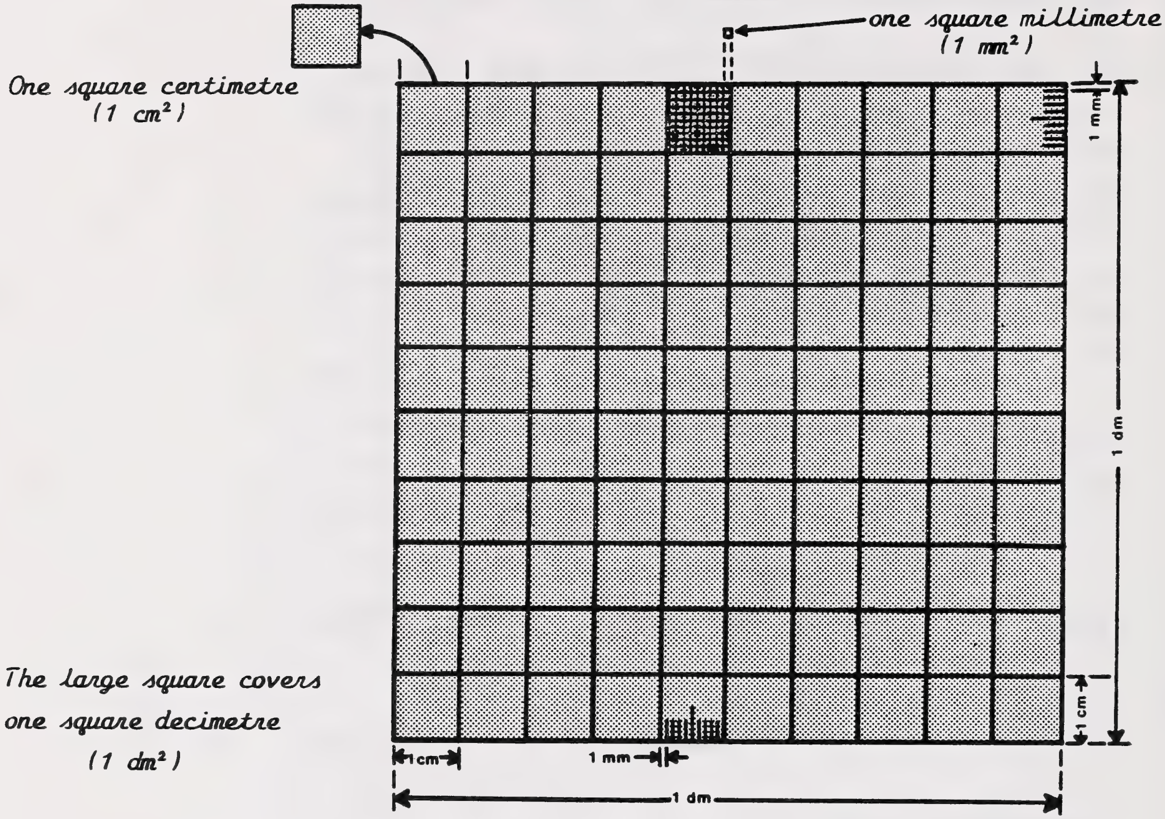
UNIT	SYMBOL
square centimetre	cm^2
square metre	m^2
square kilometre	km^2

Write the SI symbols for:

square millimetres ______
square decametres ______
square decimetres ______
square hectometres ______

Check your answers with those on page 76.

Metric Units of Area



Use the above diagram to help you fill in the blanks.

1 cm = _____ mm, so 1 cm^2 = _____ mm × _____ mm

= _____ mm^2

1 dm = _____ cm, so 1 dm^2 = _____ cm × _____ cm

= _____ cm^2

Check your answers with those on page 76.

The most commonly used square units are cm^2 and m^2. The square millimetre is too small to be very useful in daily life. The square kilometre is a large area unit used in measuring surface areas of the earth.

The acre is being replaced by the **HECTARE,** whose symbol is **"ha"**. This symbol, while not actually a part of SI, is permitted to be used with SI. The SI equivalent of the hectare is the **SQUARE HECTOMETRE** (hm^2).

1 hm = 100 m
$1\ hm^2 = (100\ m)^2 = 10000\ m^2$

$\boxed{1\ ha = 1\ hm^2}$ and $\boxed{1\ ha = 10\ 000\ m^2}$

The hectare is about the area of an average city block.

Self-correcting Exercise #4

1. **Fill in the blanks.**

1 cm = ________ mm   1 dm = ________ cm

1 cm^2 = ________ mm^2 1 dm^2 = ________ cm^2

2. **Fill in the blanks.**

1 hm^2 = 1 ha   **1 ha = 10 000 m^2**

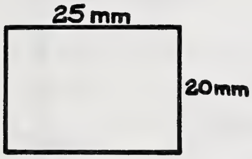
1 hm^2 = ________ m^2 50 000 m^2 = ________ ha

3 hm^2 = ________ ha   22 000 m^2 = ________ hm^2

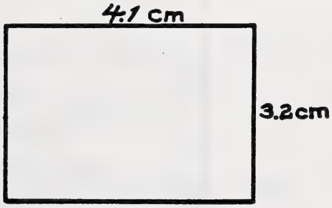
7 ha = ________ m^2 9 ha = ________ m^2

3. **Find the areas represented by the figures below. The dimensions are recorded on each shape.**

(a)



(b)

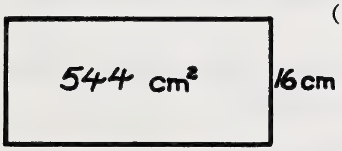


AREA = length × width
= ______ mm × ______ mm
= ________ mm^2

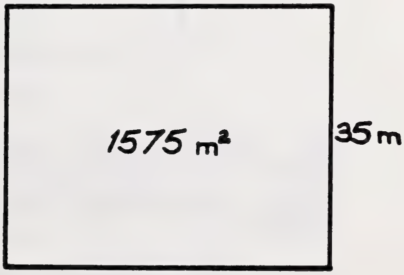
AREA = length × width
= ________________
= ________

4. **The area represented by each rectangle and its width is given below. Find the missing length.**

(a)



(b)



$$\text{Length} = \frac{\text{Area}}{\text{Width}}$$

$$= \frac{\_\_\_\_\_\_\ \text{cm}^2}{\text{cm}}$$

= ______ cm

$$\text{Length} = \frac{\text{Area}}{\text{Width}}$$

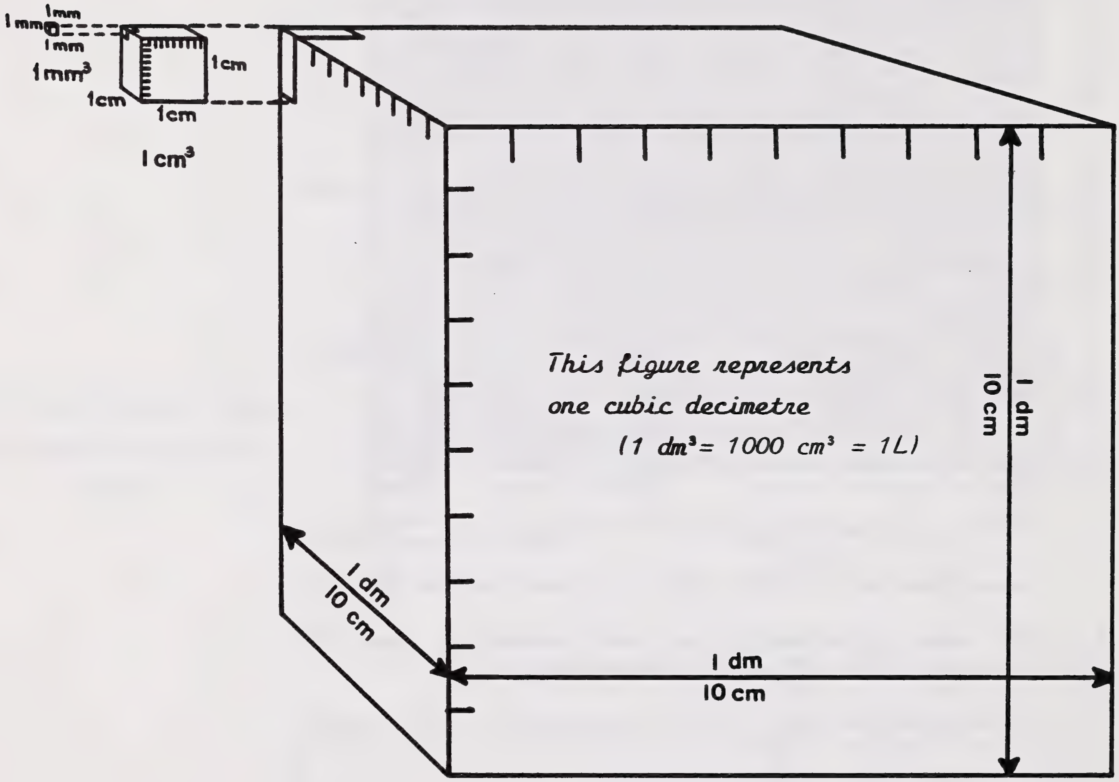
= ________

= ________

Check your answers with those on page 77.

G. Metric Units of Volume, Capacity and Mass

We know that the area of a figure is found by multiplying two dimensions (length × width). We find the volume of a figure by multiplying three dimensions (length × width × height.)



The **VOLUME** of a figure is defined as the amount of space it occupies. The measure of the volume is the number of cubic units that can be fitted into the figure.

Note that one cubic millimetre ($1\ mm^3$) is a **very** small unit while one cubic centimetre ($1\ cm^3$) is smaller than a sugar cube.

SI symbols for volume emphasize that three dimensions are involved by using the "3" as an exponent.

EXAMPLES:

UNIT	SYMBOL
cubic centimetre	cm^3
cubic metre	m^3
cubic millimetre	mm^3

Write the SI symbols for:

cubic decimetre __________ cubic hectometre __________

cubic kilometre __________ cubic decametre __________

Check your answers with those on page 77.

Self-correcting Exercise #5

Fill in the blanks.

(1) The smallest cube in the diagram on page 14 has an edge of _____ mm and its volume is _____ mm × _____ mm × _____ mm = _____ mm^3.

(2) The centimetre cube in the diagram has an edge of _____ cm or 10 mm, so its volume is _____ cm^3 or _____ mm × _____ mm × _____ mm = _____ mm^3.

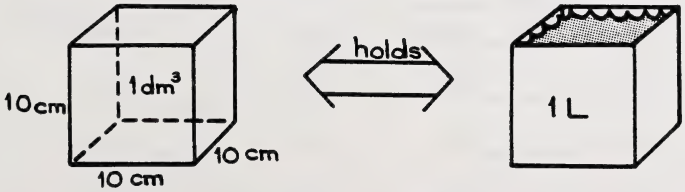
(3) The edge of the decimetre cube measures _____ dm or _____ cm so its volume is _____ dm^3 or _____ cm × _____ cm × _____ cm = _____ cm^3.

Check your answers with those on page 77.

I. What is Capacity?

The measure of the capacity of a container is the number of liquid units that it will hold. The **LITRE** is the basic unit of capacity in the metric system and the symbol for it is a capital L.

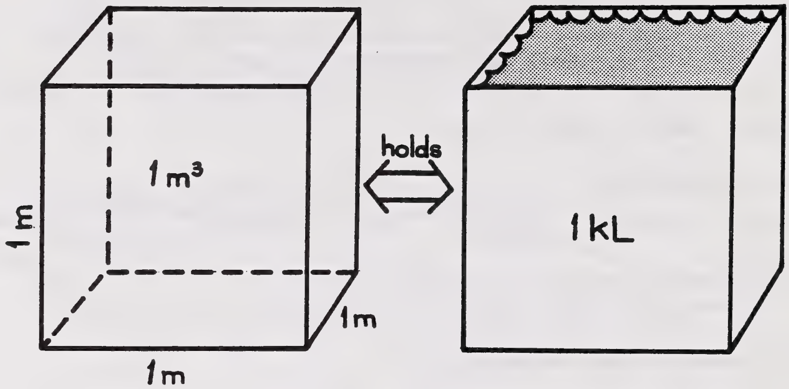
The capacity of a one decimetre cube is one litre.



$1\ dm^3 = 1\ L$

A **KILOLITRE** is a large unit of capacity. The capacity of a one metre cube is one kilolitre.

$1\ m^3 = 1\ kL$



The **most commonly** used metric units of capacity are the **litre**, the **millilitre** and the **kilolitre**.

Self-correcting Exercise #6

1. **Complete the table below.**

METRIC UNITS OF CAPACITY

kilo means thousand. 1 kL = ______ L

hecto means ______________. 1 hL = ______ L

deca means ______________. 1 daL = ______ L

deci means ______________. 1 dL = ______ L

centi means ______________. 1 cL = ______ L

milli means ______________. 1 mL = 0.001 L

2. **Complete the following. Use the table from question 1 to help you convert from one metric unit to another.**

1 kL = ______ hL

1 hL = ______ daL

1 cL = ______ dL

1 mL = ______ cL

1 kL = ______ daL

1 dL = ______ L

1 daL = ______ hL

1 mL = ______ dL

1 hL = ______ kL

1 daL = ______ kL

Check your answers with those on page 77.

II. Capacity and Volume

The capacity of a one decimetre cube is one litre. The volume of a one decimetre cube is 1 dm^3 or in cubic centimetres this volume is 10 cm × 10 cm × 10 cm = 1000 cm^3.

$1 \text{ dm}^3 = 1 \text{ L}$
$\phantom{1 \text{ dm}^3} = 1000 \text{ mL}$ and $1 \text{ dm}^3 = 1000 \text{ cm}^3$

$$1000 \text{ mL} = 1000 \text{ cm}^3$$

$$\boxed{1 \text{ mL} = 1 \text{ cm}^3}$$

Self-correcting Exercise #7

1. Fill in the blanks.

1 mL = _______ cm^3

16 mL = _______ cm^3

2.5 cm^3 = _______ mL

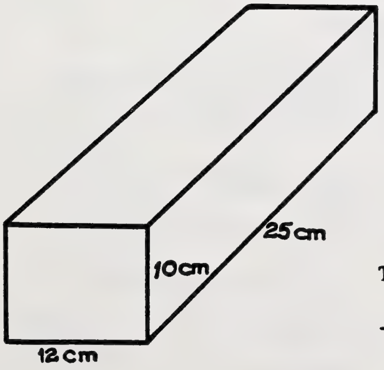
1 dm^3 = _______ L

18 dm^3 = _______ L

320 L = _______ dm^3

2. Find the volume of this box in cubic centimetres. Then give its capacity in millilitres and in litres.

Volume = _________ × _________ × _________



The capacity of the box is _______ mL or _________ L.

Check your answers with those on page 78.

III. Metric Units of Mass

MASS is defined as the amount of material in an object. The mass of an object remains the same whether the object is on earth or in space. The **WEIGHT** of an object does not remain constant but varies with the force of gravity. On earth the mass and weight of an object are almost identical.

The most important units of mass are the kilogram, the gram, and the milligram.

The **KILOGRAM** is the basic unit of mass. The standard kilogram unit is a cylinder of platinum-iridium alloy kept by the International Bureau of Weights and Measures near Paris.

A kilogram is about the mass of 10 medium-sized apples.

A litre container filled with water has a mass of one kilogram.

The **GRAM** is a small unit of mass that is about the mass of a ping-pong ball.

One millilitre of water has a mass of one gram.

The **MILLIGRAM** is a very small unit of mass measuring one-thousandth of a gram.

The **TONNE** is a large unit of mass equal to one thousand kilograms. The symbol for tonne is "t". The mass of a Volkswagen "beetle" car is about 1 t.

Self-correcting Exercise #8

1. **Complete the chart below.**

METRIC UNITS OF MASS

one kilogram (1 kg) = __________ g

one ______________ (1 hg) = __________ g

one decagram (   ) = __________ g

one ______________ (   ) = 1 g

one ______________ (   ) = 0.1 g

one centigram (   ) = __________ g

one ______________ (1 mg) = __________ g

Self-correcting Exercise #9

Fill in the blanks.

(1) 1 cm^3 of water is equivalent to _______ mL and weighs _______ g.

(2) _______ cm^3 of water are equivalent to 95 mL and weigh _______ g.

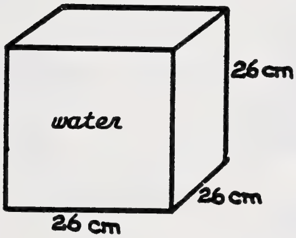
(3) 1 dm^3 of water is equal to _______ L and weighs _______ kg.

(4) One cubic metre has a capacity of _______ kL or _______ L.

(5) The weight of 1 kL of water is _______ t.

(6) 10.5 m^3 of water is equivalent to _______ kL and weigh _______ t.

(7) Find the weight of water in grams and in kilograms that can be held by the container below.



Volume = ________ × ________ × ________

= _______ cm^3

= _______ mL

The weight of the water in grams is _______ g and in kilograms is _______ kg.

Check your answers with those on page 78.

Exercise - Area, Volume, Mass and Capacity

1. **Fill in the blanks.**

Note:

1 hm = 100 m
1 hm^2 = 10000 m^2
1 hm^2 = 1 ha

7.2 L = _______ dm^3
632 g = _______ kg
2 hm^2 = _______ m^2
121 cm^3 = _______ mL
75 hm^2 = _______ ha
35 kg = _______ g
50 000 mg = _______ g

83.7 g = _______ mg
10 000 g = _______ kg
2 550 mg = _______ g
323.6 g = _______ mg
10 dm^3 = _______ L
920 kg = _______ g
15 ha = _______ m^2

2. **Complete the following statements. Decide whether to multiply by 1000 or by 0.001.**

1 kg = __________ g

7 kg = __________ g

12.2 kg = __________ g

1 g = __________ kg

3 g = __________ kg

52.5 g = __________ kg

3. **Change gram measures to milligram measures and vice-versa.**

1 g = __________ mg

5 g = __________ mg

43 g = __________ mg

1 mg = __________ g

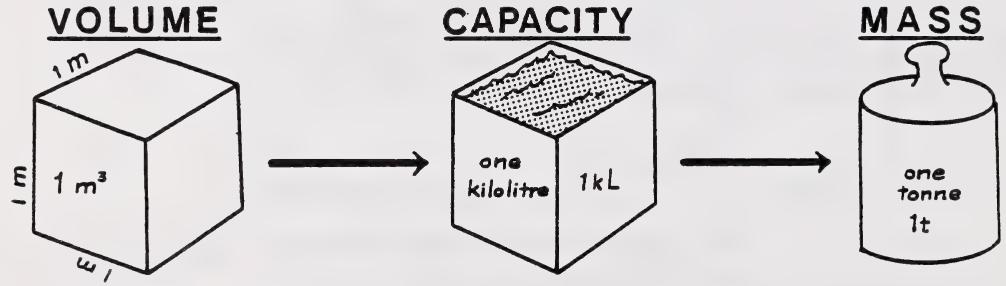
13 mg = __________ g

64.5 mg = __________ g

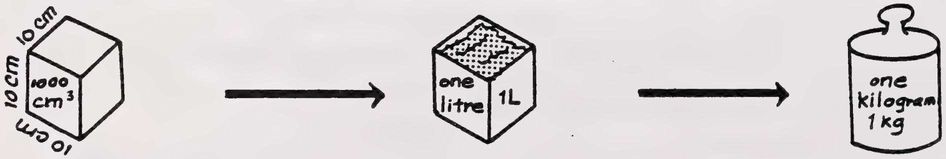
Check your answers with those on page 78.

IV. Units of Volume, Capacity and Mass

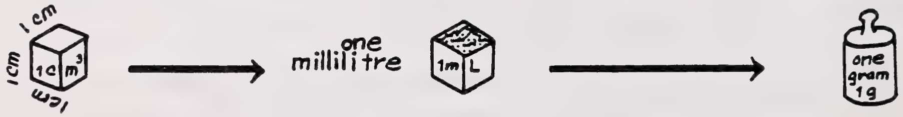
One reason why the metric system is easy to use is the direct relationship between units of volume, capacity and mass. This special relationship is illustrated in the chart below. **MEMORIZE** these facts.



A cube whose volume is 1 m^3 holds 1 kL of water and weighs 1 t.



A cube whose volume is 1000 cm^3 (1 dm^3) holds 1 L of water and weighs 1 kg.



A cube whose volume is 1 cm^3 holds 1 mL of water and weighs 1 g.

2. Using a ruler, draw the figures specified. Record the dimensions and area for each figure.

(a) A square 4 cm to a side.

(b) A rectangle 5 cm by 2.5 cm.

(a)

Length = ________ cm

Width = ________ cm

Area = Length × Width

= ______ cm × ______ cm

= ______ cm^2

(b)

Length = ____________

Width = ____________

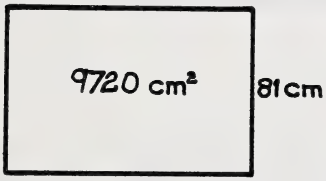
Area = Length × Width

= ______________________

= ______________________

3. Given the area of each rectangle below, find the missing dimension.

(a) Find the length.

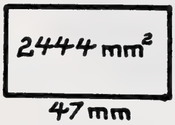


Length = $\frac{\text{Area}}{\text{Width}}$

= ________

= ________

(b) Find the width.



Width = $\frac{\text{Area}}{\text{Length}}$

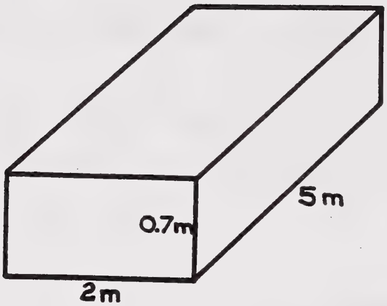
= ________

= ________

4. **Find the volume of a tank in cubic metres. Then give its capacity in litres and in kilolitres.**

Volume = _______ × _______ × _______

= _______________

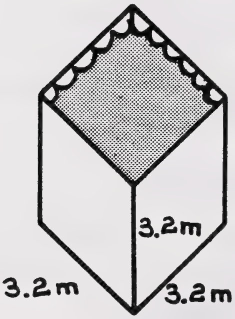


The capacity of the tank is _______ L or _______ kL.

5. **Find the weight in tonnes and in kilograms of the water which could be held in a tank measuring 3.2 m × 3.2 m × 3.2 m.**

Volume = _______ × _______ × _______

= _______________



The capacity of the tank is _______ t or _______ kg.

TOPIC TWO: TRIGONOMETRY

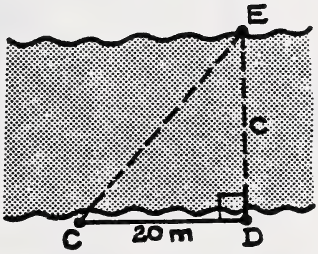
A. Introduction

Trigonometry is the branch of mathematics which deals with the use of triangles for indirect measurements of distances, lengths and heights.

Distances which cannot be measured directly can be determined by making scale drawings.

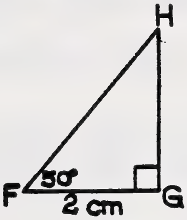
EXAMPLE:

To calculate the distance across the river in the diagram at the right, we could construct a scale drawing, and by measuring the corresponding sides and angles, we could calculate side c.



DE is the distance across the river.

Using a scale of 1 cm = 10 m, we can draw FG = 2 cm, which corresponds to CD in ΔCDE. Then at point F, we construct ∠F = 50° and at point G, we construct ∠G = 90°. The arms of these angles intersect at point H.



ΔFGH is a scale drawing of ΔCDE. We can measure HG and apply the scale factor of 1 cm = 10 m to determine length c in ΔCDE.

$$\text{i.e.}\quad c = 10 \times HG^*$$

$$= 10 \times 2.4$$

$$= 24 \text{ m}$$

$\overline{HG}$ measures 2.4 cm.

However, drawing scale diagrams is not always convenient.

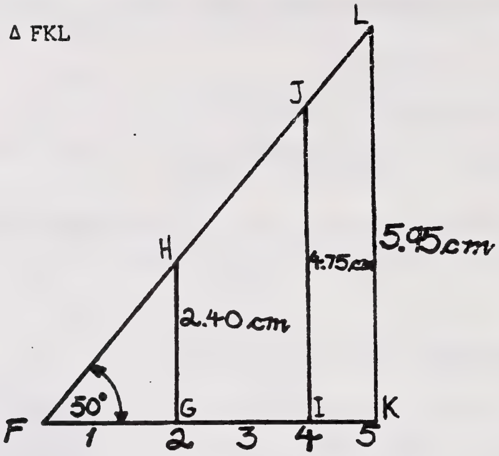
By constructing triangles similar to $\triangle CDE$ we have

"is similar to"

$$\triangle FGH \sim \triangle FIJ \sim \triangle FKL$$

and

$$\frac{HG}{FG} = \frac{JI}{FI} = \frac{LK}{FK}$$



Using the measurements given in the diagram we can calculate the ratios above.

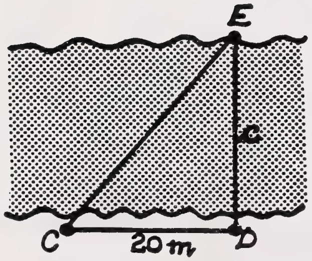
$$\frac{HG}{FG} = \frac{2.40}{2} = 1.2$$

$$\frac{JI}{FI} = \frac{4.75}{4} = \underline{1.1875}$$

$$\frac{LK}{FK} = \frac{5.95}{5} = \underline{1.19}$$

The three ratios are approximately the same.

If we take the average of the above ratios we get 1.1925. We can use this ratio to calculate the distance across the river. This ratio is called the tangent of the angle F.



In $\triangle CDE, \frac{ED}{CD} = 1.1925$ or $\frac{c}{CD} = 1.1925$

CD = 20m

$$\frac{c}{20} = 1.1925$$

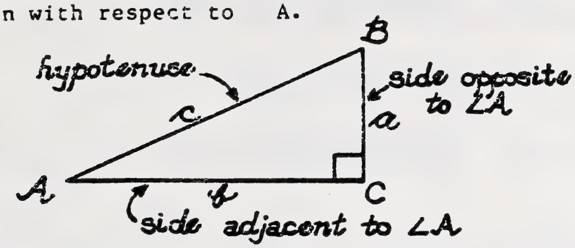
$$\begin{aligned} c &= 20 \times 1.1925 \\ &= 23.85 \\ &= 23.9 \text{ (to 3 sig. digits.)}^{*} \end{aligned}$$

In general, for any right triangle containing a 50° angle, the tangent ratio with respect to this angle will always be the same.

**The answer is given to 3 significant digits since the original measurements were made to 3 significant digits.*

B. Primary Ratios of Trigonometry

In the right-angled triangle below, names for the sides are given with respect to A.



In a right triangle, the "Hypotenuse" is always the side opposite the right angle.

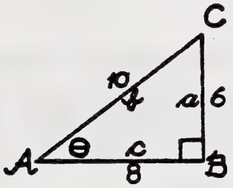
For ∠A, we define the following ratios:

$$\boxed{\begin{aligned} \text{sine of } \angle A &= \frac{\text{side opposite to } \angle A}{\text{hypotenuse}} = \frac{a}{c} \\ \text{cosine of } \angle A &= \frac{\text{side adjacent to } \angle A}{\text{hypotenuse}} = \frac{b}{c} \\ \text{tangent of } \angle A &= \frac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \frac{a}{b} \end{aligned}}$$

We can write the above ratios in abbreviated form as follows:

$$\boxed{\sin A = \frac{a}{c},\ \cos A = \frac{b}{c},\ \tan A = \frac{a}{b}}$$

e.g.



We can write

$\sin\theta = \frac{6}{10}$, $\cos\theta = \frac{8}{10}$,

and $\tan\theta = \frac{6}{8}$.

The Greek letter "θ" (pronounced thay-ta) is used to name the unknown angle.

If we know the measures of two sides of a right triangle we can find the three trigonometric ratios associated with the two acute angles in the triangle.

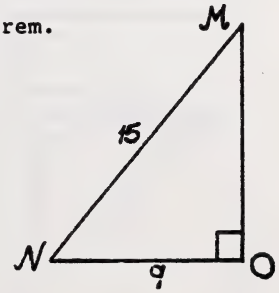
EXAMPLE:

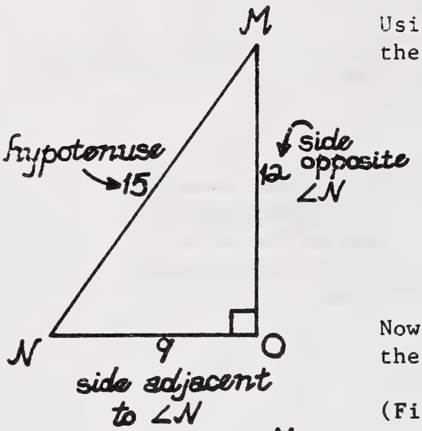
In Δ MNO find the primary trigonometric ratios for ∠M and ∠N.

Solution:

Find the measure of the third side.
Calculate MO using the Pythagorean Theorem.

$$\begin{aligned} 15^2 &= MO^2 + 9^2 \\ MO^2 &= 15^2 - 9^2 \\ &= 225 - 81 \\ MO^2 &= 144 \\ MO &= 12 \end{aligned}$$





Using the definitions on page 25, we can determine the trigonometric ratios associated with ∠N.

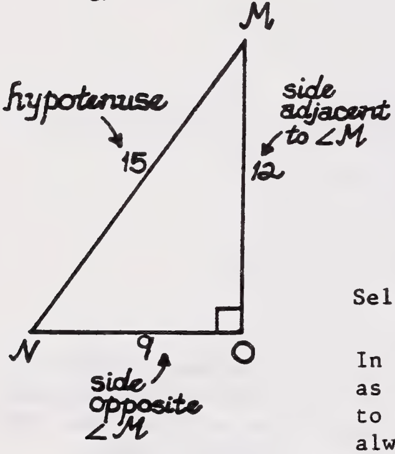
$$\sin \angle N = \frac{\text{side opposite to } \angle N}{\text{hypotenuse}} = \frac{12}{15}$$

$$\cos \angle N = \frac{\text{side adjacent to } \angle N}{\text{hypotenuse}} = \frac{9}{15}$$

$$\tan \angle N = \frac{\text{side opposite to } \angle N}{\text{side adjacent to } \angle N} = \frac{12}{9}$$

Now, we can apply the same definitions to determine the trigonometric ratios associate with ∠M.

(Fill in the blanks below.)



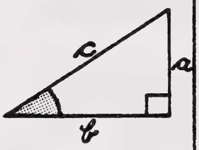
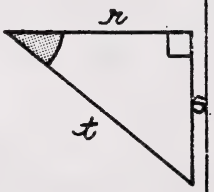
$$\sin \angle M = \frac{\text{side opposite to } \angle M}{\text{hypotenuse}} = \frac{\quad}{15}$$

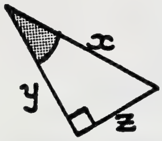
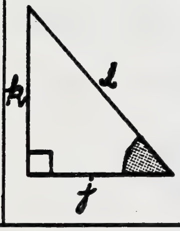
$$\cos \angle M = \underline{\hspace{6cm}} = \frac{MO}{MN} = \text{—}$$

$$\tan \angle M = \underline{\hspace{6cm}} = \frac{\quad}{MO} = \text{—}$$

Self-Correcting Exercise #10

In each of the given triangles, identify the sides as the hypotenuse, side opposite or side adjacent to the indicated angle. (Remember that the hypotenuse always lies across from the right angle.)

Triangle	Hypotenuse	Side Opposite	Side Adjacent
1. 	c	a	b
2. 			

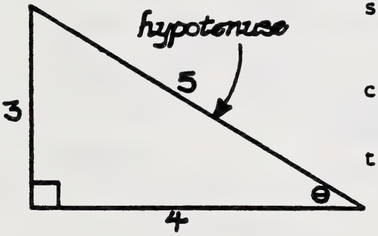
Triangle	Hypotenuse	Side Opposite	Side Adjacent
3. 			
4. 			

Check your answers with those on page 79.

Exercise: Primary Trigonometric Ratios

1. **State the value of (i) sin θ (ii) cos θ (iii) tan θ for each given triangle.**

(a)

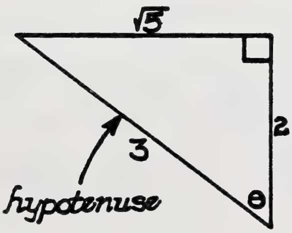


$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{3}{5}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \underline{\qquad}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \underline{\qquad}$

(b)

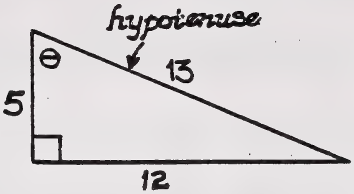


$\sin\theta = \dfrac{\underline{\qquad}}{\text{hyp}} = \underline{\qquad}$

$\cos\theta = \underline{\qquad} = \underline{\qquad}$

$\tan\theta = \underline{\qquad} = \underline{\qquad}$

(c)



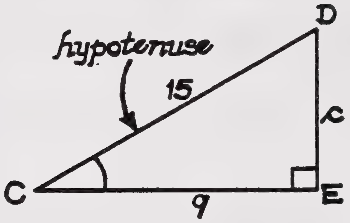
$\sin\theta = \dfrac{\underline{\qquad}}{\text{hyp}} = \dfrac{\underline{\qquad}}{13}$

$\cos\theta = \underline{\qquad} = \underline{\qquad}$

$\tan\theta = \underline{\qquad} = \underline{\qquad}$

2.

(a) Find the sine, cosine, and tangent ratio for ∠C. (Hint: Use the Pythagorean Theorem to find the unknown side.)



$15^2 = 9^2 + c^2$

$c^2 = 15^2 - 9^2$

$c^2 = \underline{\qquad} - \underline{\qquad}$

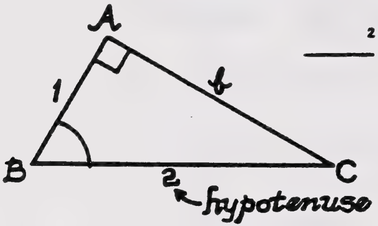
$c^2 = \underline{\qquad}$

$c = \underline{\qquad}$

$\sin C = \dfrac{\text{opp}}{\text{hyp}} = \underline{\qquad}$, $\cos C = \dfrac{\text{adj}}{\text{hyp}} = \underline{\qquad}$.

$\tan C = \dfrac{\text{opp}}{\text{adj}} = \underline{\qquad}$

(b) Find the sine, cosine, tangent ratios for ∠B.



$\underline{\qquad}^2 = \underline{\qquad}^2 + \underline{\qquad}^2$

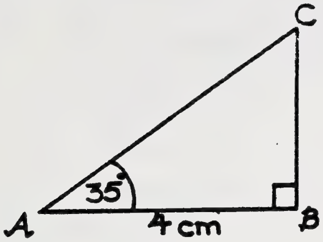
$\sin B = \underline{\qquad} = \underline{\qquad}$, $\cos B = \underline{\qquad} = \underline{\qquad}$,

$\tan B = \underline{\qquad} = \underline{\qquad}$

C. Trigonometric Ratios for Any Acute Angle

We can find the trigonometric ratios associated with any acute angle by constructing a right triangle containing that angle. After measuring the three sides of the triangle, we can apply definitions of sine, cosine, and tangent in order to find the trigonometric ratios.

EXAMPLE - Find the trigonometric ratios for 35°.



Solution

Construct $\triangle ABC$ with $\angle A = 35^\circ$, $AB = 4\text{cm}$ and $\angle B = 90^\circ$. (Note that any length can be used for $\overline{AB}$.)

Measuring the other two sides of $\triangle ABC$, we can find that:

$AC = 4.9\text{cm}$ and $BC = 2.8\text{cm}$. (to 2 sig. digits)

Applying the definitions of sin, cos, and tan we find that:

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AC} = \frac{2.8}{4.9} = 0.57 \quad \text{(to 2 sig. digits)*}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AB}{AC} = \underline{\hspace{2cm}} = \underline{\hspace{2cm}} \quad \text{(to 2 sig. digits)}$$

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AB} = \underline{\hspace{2cm}} = \underline{\hspace{2cm}} \quad \text{(to 2 sig. digits.)}$$

**We have divided 2.8 by 4.9 to obtain a 3-digit answer and then rounded to 2 digits. (Answers shouldn't have more significant digits than the original measurements.)*

Fortunately, we do not require diagrams each time we need a trigonometric ratio. The table on page 85 of this lesson gives the sin, cos, and tan values of the 0° to 90°. Note that the values you calculated in the above example are close to the values for 35° in the table. (Table values are more accurate because they are determined by a computer.) Trigonometric tables can be used to find the sin, cos, and tan of a particular angle or to find the measure of an angle given one of its trigonometric ratios.

EXAMPLE:

(a) Find tan 18°.
(b) If $\sin \theta = 0.4848$, find θ.

Solution

(a) From page 85, under the tan column for 18°, the value is 0.3249. So $\tan 18^\circ = 0.3249$.

(b) In the sin column look for the decimal value of 0.484. If $\sin \theta = 0.4848$, then θ is 29°.

Self-Correcting Exercise #11

1. Using the table on page 85 of this lesson, find values for the following angle measures.

(a) $\sin 20^\circ$ ______ (b) $\tan 86^\circ$ ______

(c) $\cos 78^\circ$ ______ (d) $\sin 90^\circ$ ______

(e) $\cos 0^\circ$ ______ (f) $\tan 0^\circ$ ______

2. Using the table, find the angle measures for the following trigonometric ratios.

(a) $\sin \theta = 0.5299$ (b) $\tan \theta = 1.963$

$\theta =$ ______ $\theta =$ ______

(c) $\sin \theta = 0.9998$ (d) $\cos \theta = 0.3256$

$\theta =$ ______ $\theta =$ ______

Check your answers with those on page 79.

Sometimes you may be given a trigonometric ratio which does not occur in the table. You must then determine which value in the table is closest to the given value.

EXAMPLE:

Find the angle θ for $\sin \theta = 0.4860$

Solution

Looking in the table on page 85 we find:

$$\begin{aligned} \sin 29^\circ &= .4848 \\ \sin 30^\circ &= .5000 \end{aligned}$$

0.4860 lies between these values.

We find the absolute difference between the given value and each of the two values from the table.

$$\begin{array}{r} 0.4860 \\ -0.4848 \\ \hline 0.0012 \end{array} \qquad \qquad \begin{array}{r} 0.5000 \\ -0.4860 \\ \hline 0.0140 \end{array}$$

given values

smaller difference

Since $0.0012 < 0.0140$, then 0.4848 is closer to the given value.

Thus, $0.4860 \doteq \sin 29$ to the nearer degree.

i.e. $\theta \doteq 29^\circ$

If the absolute differences are the same, then round to the higher degree.

Self-correcting Exercise #12

Use the table on page 85 to find each angle to the nearer degree. (Space for rough work has been provided.)

1. $\cos\theta = 0.9850$

   $\theta =$ ______

   Rough work:

2. $\cos\theta = 0.5055$

   $\theta =$ ______

   Rough work:

3. $\sin\theta = 0.7500$

   $\theta =$ ______

   Rough work:

4. $\sin\theta = 0.9999$

   $\theta =$ ______

   Rough work:

5. $\tan\theta = 0.0500$

   $\theta =$ ______

   Rough work:

6. $\cos\theta = 0.2000$

   $\theta =$ ______

   Rough work:

Check your answers with those on pages 80 and 81.

EXERCISE: Trigonometric Ratios

1. From the table on page 85, find the values for the following angle measures.

(a) tan 2° = ________ (b) cos 60° = ________

(c) sin 60° = ________ (d) tan 45° = ________

(e) sin 30° = ________ (f) cos 21° = ________

(g) cos 49° = ________ (h) sin 82° = ________

2. From the table on page 85 find the angle measures for the following trigonometric ratios. Find the angle to the nearer degree where necessary.

(a) $\cos\theta = 0.7071$ (b) $\tan\theta = 57.29$

θ = ______ θ = ______

(c) $\cos\theta = 1.0000$ (d) $\sin\theta = 0.7071$

θ = ______ θ = ______

(e) $\sin\theta = 0.9333$ (f) $\tan\theta = 0.8525$

θ = ______ θ = ______

Rough work: Rough work:

(g) $\sin\theta = 0.9691$ (h) $\tan\theta = 0.4444$

θ = ______ θ = ______

Rough work: Rough work:

3. Examine the trigonometic tables at the end of the lesson and complete the folllowing statements.

   (a) As the angle θ , increases from 0° to 90°, sin θ increases from __________ to __________.

   (b) As the angle θ , increases from 0° to 90°, cos θ decreases from __________ to __________.

   (c) As the angle θ , increases from 0° to 89°, tan θ __________ from __________ to __________.

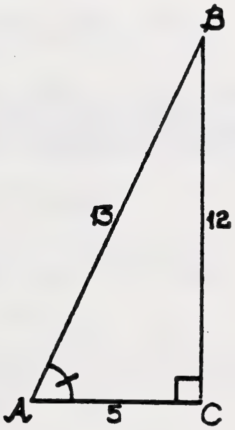
4. Determine the functions given the following ratios.

   (a) For ∠A,

      (i) $\frac{12}{5}$ = tan A

      (ii) $\frac{12}{13}$ = ______

      (iii) $\frac{5}{13}$ = ______



   (b) For ∠B.

      (i) $\frac{12}{13}$ = ______

      (ii) $\frac{5}{12}$ = ______

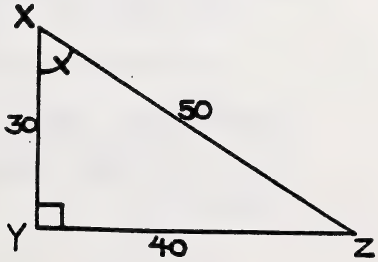
      (iii) $\frac{5}{13}$ = ______

   (c) For ∠X,

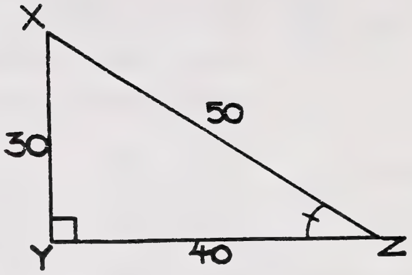
      (i) $\frac{30}{50}$ = ______

      (ii) $\frac{40}{30}$ = ______

      (iii) $\frac{40}{50}$ = ______



(d) For ∠Z



(i) $\frac{30}{40}$ = ________

(ii) $\frac{40}{50}$ = ________

(iii) $\frac{30}{50}$ = ________

D. Solving Right Triangles

A triangle is solved when the measures of all sides and of all angles have been determined. A right triangle can be solved, provided we are given any of the following:

1. the measures of two sides;
2. the measure of one side and the measure of one angle other than the right angle.

Techniques used for the solution vary depending on the information given. Any of the following may be used in solving a right triangle:

1. trigonometric ratios;
2. the table of trigonometric ratios;
3. the fact that the sum of the measures of the angles in any triangle is 180°;
4. the Pythagorean Theorem.

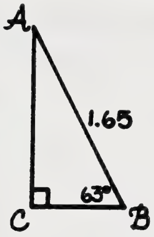
Please note: Whenever possible avoid finding measures using calculated measures. If a calculated measure is wrong, any other measures calculated using it will be wrong also.

EXAMPLE 1:

In $\triangle ABC$, if $\angle B = 63^\circ$ and $AB = 1.65$, solve the triangle. $\angle C$ is the right angle. (Express answers to 2 decimal places.)

SOLUTION:

We begin by drawing a sketch of the triangle, marking the given measures on the triangle.



Solve for angle A first.

$$\angle A = 180^\circ - (\angle B + \angle C)$$
$$= 180^\circ - (63^\circ + 90^\circ)$$
$$= 180^\circ - 153^\circ$$
$$= 27^\circ$$

Trigonometric ratios can now be used to solve for AC and CB. When solving for AC, we have the side opposite the 63° angle and the hypotenuse in the ratio, so the sine function should be used.

$$\sin 63^\circ = \frac{AC}{1.65}$$
$$AC = \sin 63^\circ \times 1.65$$
$$AC = 0.8910 \times 1.65$$
$$AC = 1.47015$$
$$AC = 1.47$$

To solve for CB, we need to find side adjacent to the 63° while the measure of the hypotenuse is given. Thus, we must use the cosine ratio.

Complete this solution:

$\cos 63^\circ =$

Now, we have solved for $\triangle ABC$.

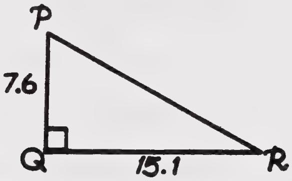
$\angle A = 27°$ $AB = 1.65$

$\angle B = 63°$ $CB = 0.75$

$\angle C = 90°$ $AC = 1.47$

EXAMPLE 2:

Solve the triangle pictured below.



SOLUTION:

We can use the Pythagorean Theorem to solve for PR.

$$PR^2 = 7.6^2 + 15.1^2$$

$$PR^2 = 57.76 + 228.01$$

$$PR^2 = 285.77$$

$$PR = \sqrt{285.77}$$

$$PR = 16.9$$

We avoid using this measurement in establishing measures for $\angle P$ or $\angle R$, since any error in the calculations of PR would alter the values of $\angle P$ or $\angle R$.

To find $\angle P$, we use the tangent function.

$$\tan P = \frac{15.1}{7.6}$$

$$= 1.9868$$

$$\angle P = 63°$$

To find $\angle R$, we also use the tangent function.

$$\tan R = \frac{7.6}{15.1}$$

$$= 0.5033$$

$$\angle R = 27°$$

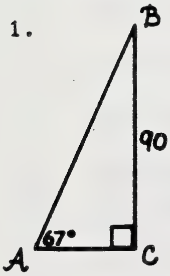
Thus, our triangle is solved.

$\angle P = 63^\circ$ $PQ = 7.6$
$\angle R = 27^\circ$ $QR = 15.1$
$\angle Q = 90^\circ$ $PR = 16.9$

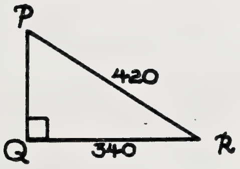
EXERCISE —Solving Right Triangles

Use the information given on each diagram to completely solve each triangle.

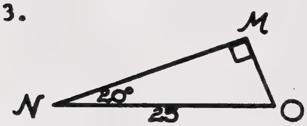
1.



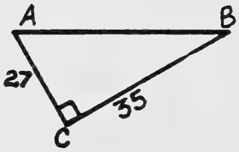
2.



3.



4.



E. Using Trigonometry to Solve Problems

When using trigonometry to solve problems, it is best to use an organized approach. To assist in your solutions, follow the 7-step method presented below.

7-STEP PROBLEM SOLVING METHOD

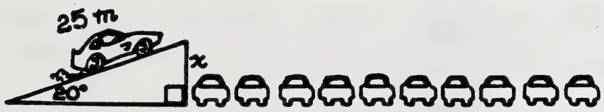
1. **Read the problem carefully.**
2. **Draw a diagram which represents the problem.**
3. **Mark all known values on your diagram.**
4. **Mark the unknown part(s) you will need to find.**
5. **Determine whether you need to use the sine, cosine or tangent ratio for your solution. (Select one in which two values are known and only one value is unknown.)**
6. **State the trigonometric ratio you will be using and solve for the unknown part.**
7. **Make a final statement which relates the answers to the question.**

You may need to repeat steps 5 and 6 if you have to solve for more than one unknown part.

EXAMPLE 1:

The ramp built for a jump over 10 cars is inclined at 20°. If the ramp is 25 m long, how far above the ground will the car be when it leaves the ramp? (Express your answer to one decimal place.)

SOLUTION:



Steps 2, 3 and 4

Step 5

We know the angle and the hypotenuse. We are trying to find the side opposite the angle. Thus, we need to use the sine ratio.

$$\sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}}$$

Step 6

$$\sin 20^\circ = \frac{x}{25}$$

$$x = \sin 20^\circ \times 25$$

$$x = 0.3420 \times 25$$

$$x = 8.55$$

$$x = 8.6 \text{ m (to one decimal place)}$$

Step 7

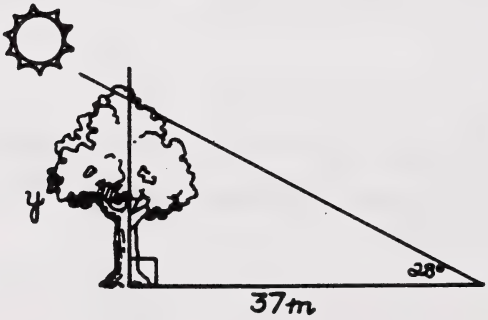
The car will be 8.6 m above the ground when it leaves the ramp.

EXAMPLE 2:

Find the height of a tree if it has a shadow of 37 m and the angle which the sun's rays make with the ground is 28°. (Round your answer to the nearer whole metre.)

SOLUTION:

Steps 2, 3, and 4



Step 5

We know the angle and the side adjacent to the angle. We need to know the side opposite the angle. Thus, the tangent function must be used.

$$\tan \theta = \frac{\text{side opposite}}{\text{side adjacent}}$$

$$\begin{aligned} \tan 28^\circ &= \frac{y}{37} \\ y &= \tan 28^\circ \times 37 \\ &= 0.5317 \times 37 \\ &= 19.67 \\ &= 20 \text{ m (to nearer whole metre)} \end{aligned}$$

Step 6

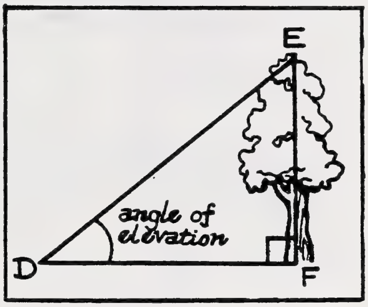
The tree is approximately 20 m tall.

Step 7

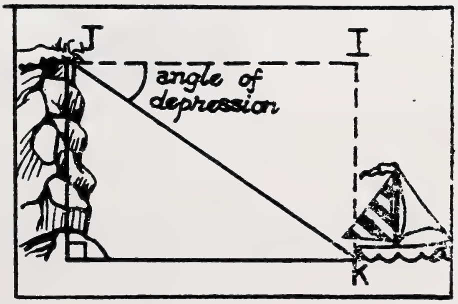
Angles of Elevation and Depression

In using trigonometry to solve problems, you will often encounter the terms "angle of elevation" and "angle of depression. Study the following diagrams which should help clarify the meanings of these terms.

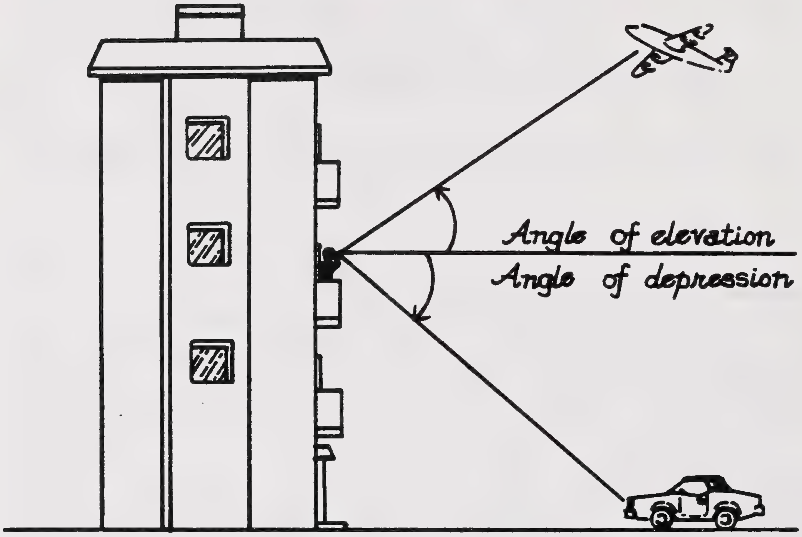
∠EDF is called an angle of elevation since it is measured **upward** from the horizontal line, DF.



∠IJK is called an angle of **depression** since it is measured **downward** from the horizontal line, IJ.



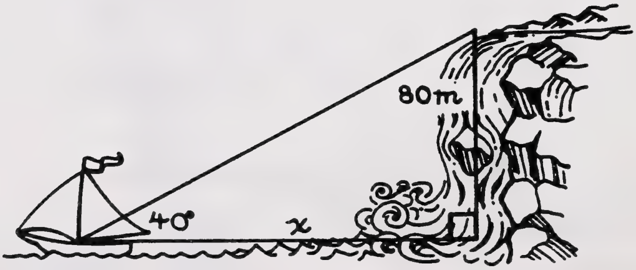
To relate the two types of angles, consider a person in a high-rise building. He would form an angle of elevation looking up to a plane in the sky and form an angle of depression looking down to a car in the street.



EXAMPLE 1:

The angle of elevation of the top of a waterfall from a boat is 40°. If the waterfall is known to be 80 m high, how far from the waterfall is the boat?

SOLUTION:



Since we know the side opposite the 40° angle and we want to find the side adjacent to the 40° angle, the tangent is the function which must be used.

$$\tan 40^\circ = \frac{80}{x}$$

$$x = \frac{80}{\tan 40^\circ}$$

$$x = \frac{80}{0.8391}$$

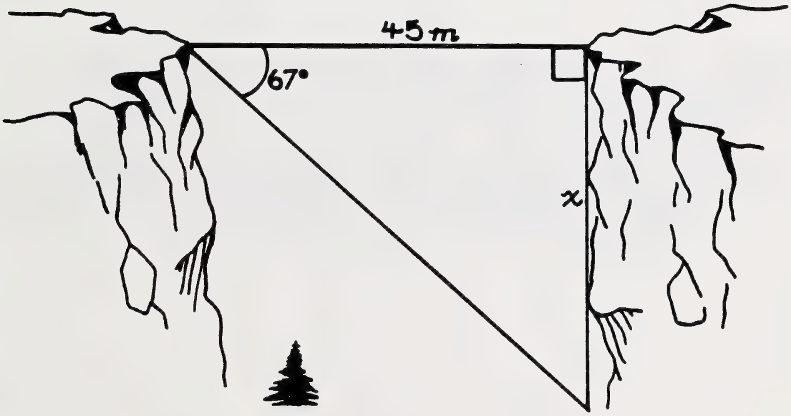
$$x = 95 \text{ m}$$

The boat is approximately 95 m from the waterfall.

EXAMPLE 2:

How deep is a gorge if the angle of depression of the bottom at the opposite side is 67° and the distance across the gorge is 45 m?

SOLUTION:



Since we know the side adjacent to the 67° angle and want to know the side opposite the 67° angle, we need to use the tangent function.

$$\tan 67^\circ = \frac{x}{45}$$

$$x = \tan 67^\circ \times 45$$

$$= 2.3559 \times 45$$

$$= 106 \text{ m}$$

The gorge is about 106 m deep.

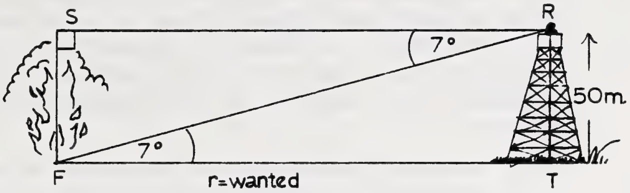
Self-Correcting Exercise #13

Solve the following problems using the 7-step method outlined on page 39. Express your answers to one decimal place.

1. A tree which is 30 m tall casts a shadow which is 70 m long. What is the angle of the sun?

2. The angle of elevation of the top of a building from a point 60 m from its base is 23°. What is the height of the building? (The building has a flat roof.)

3. The forest ranger spotted a fire from atop the fire lookout. If the angle of depression of the fire is 7° and the tower is 50 m high, how far away was the fire?



Note:

∠SRF is the angle of depression and it equals 7°. ∠RFT is the angle of elevation and it must also equal 7°. (i.e. the angle of depression and the angle of elevation are equal since they are corresponding angles of the congruent triangles RSF and FTR.)

$$\tan 7° = \frac{RT}{FT}$$

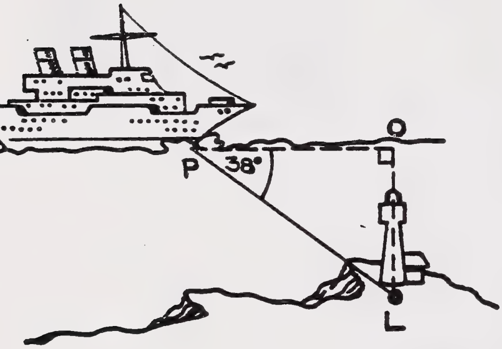
4. A hot air balloonist was 300 m directly above a farmhouse. This farmhouse is the middle one of three farmhouses which form a straight line. If his angles of depression of the first and last farmhouses are 16° and 40° respectively, what is the distance from the first farmhouse to the last farmhouse?

Check your answers with those given on pages 81 and 82.

EXERCISE: Solving Problems with Trigonometry

1. A ship is travelling along the coast at the land speed of 10 km per hour. At point P a sighting of a lighthouse is made and found to be 38°. In exactly 2 minutes the ship is directly opposite the lighthouse. Calculate how close the ship comes to the lighthouse. (Express your answer to the nearer hundredth of a kilometre.)

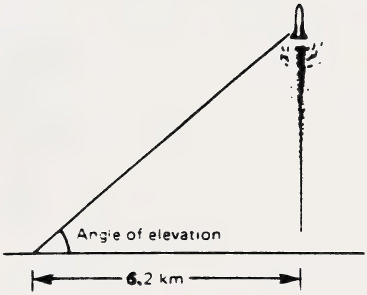
   Hint: Convert 2 minutes to a fraction of an hour, then calculate PO using speed and time.



2. A tree 15 m high casts a shadow 27 m long. Calculate the angle of elevation of the sun. (Give your answer to the nearer degree.)

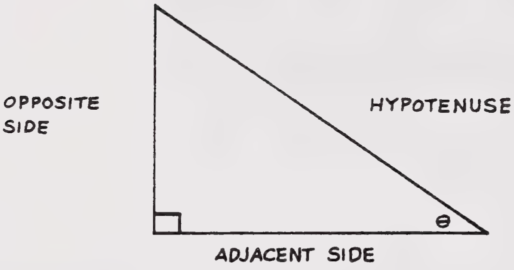
3. From a cliff, the angle of depression of a boat is 28°. If the cliff is 75 m high, how far away is the boat from the base of the cliff? (Express your answer to the nearer whole metre.)

4. An observer 6.2 km from the launch pad observed a missile ascending. At a particular time, the angle of elevation is 32°. How far is the missile from the observer?



F. The Reciprocal Trigonometric Ratios

Considering again our right triangle with angle θ :



We define the reciprocal trigonometric ratios as follows:

$$\text{cosecant of } \theta = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\text{secant of } \theta = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\text{cotangent of } \theta = \frac{\text{adjacent side}}{\text{opposite side}}$$

These ratios are abbreviated as follows:

$$\boxed{\begin{aligned} \csc\theta &= \frac{\text{hyp}}{\text{opp}} \\ \sec\theta &= \frac{\text{hyp}}{\text{adj}} \\ \cot\theta &= \frac{\text{adj}}{\text{opp}} \end{aligned}}$$

We refer to these as the reciprocal trigonometric ratios, since the cosecant is the reciprocal of the sine, the secant is the reciprocal of the cosine, and the cotangent is the reciprocal of the tangent.

That is,

$$\csc\theta \times \sin\theta = \frac{\overset{1}{\cancel{\text{hyp}}}}{\underset{1}{\cancel{\text{opp}}}} \times \frac{\overset{1}{\cancel{\text{opp}}}}{\underset{1}{\cancel{\text{hyp}}}}$$

$$= 1$$

$$\sec\theta \times \cos\theta = \frac{\overset{1}{\cancel{hyp}}}{\underset{1}{\cancel{adj}}} \times \frac{\overset{1}{\cancel{adj}}}{\underset{1}{\cancel{hyp}}}$$

$$= 1$$

$$\cot\theta \times \tan\theta = \frac{\overset{1}{\cancel{adj}}}{\underset{1}{\cancel{opp}}} \times \frac{\overset{1}{\cancel{opp}}}{\underset{1}{\cancel{adj}}}$$

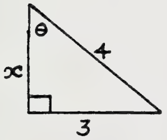
$$= 1$$

We can now identify all six trigonometric ratios using the ratios of the sides of a right triangle.

EXAMPLE:

If $\sin\theta = \frac{3}{4}$, find the remaining five trigonometric ratios.

SOLUTION:



We first draw a sketch of a right triangle on which we show the angle, θ, the side opposite θ as 3 and the hypotenuse as 4. We must apply the Pythagorean Theorem to find the measure of the side adjacent to θ before forming other ratios.

$$3^2 + x^2 = 4^2$$

$$9 + x^2 = 16$$

$$x^2 = 7$$

$$x = \sqrt{7}$$

Now the remaining ratios can be read from the diagram.

$\cos\theta = \frac{\sqrt{7}}{4}$

$\csc\theta = \frac{4}{3}$

$\sec\theta = \frac{4}{\sqrt{7}}$

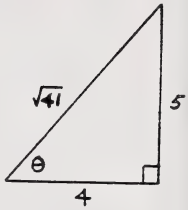
$\tan\theta = \frac{3}{\sqrt{7}}$

$\cot\theta = \frac{\sqrt{7}}{3}$

Self-Correcting Exercise #14

1. For each triangle given below, state the six trigonometric ratios for the indicated angle.

(a)

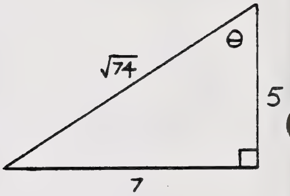


$\sin\theta$ = ______ $\csc\theta$ = ______

$\cos\theta$ = ______ $\sec\theta$ = ______

$\tan\theta$ = ______ $\cot\theta$ = ______

(b)



$\sin\theta$ = ______ $\csc\theta$ = ______

$\cos\theta$ = ______ $\sec\theta$ = ______

$\tan\theta$ = ______ $\cot\theta$ = ______

2. When $\tan\theta = \frac{6}{10}$, find the remaining five trigonometric ratios.

Check your answers with those given on page 83 of this lesson.

G. Using Tables to Find Reciprocal Trigonometric Values

The technique used for looking up values for reciprocal trigonometric ratios is the same as for the basic sine, cosine and tangent ratios. The only difference is that the tables for cosecant, secant and cotangent on page 86 must be used.

EXAMPLE:

Use the tables to find values for sec 56°, cot 65° and csc48°.

SOLUTION:

The following table is taken from page 86.

Cosecants, Secants and Cotangents 45° – 90°

Angle	Cosecant	Secant	Cotangent
45°	1.414	1.414	1.0000
46°	1.390	1.440	0.9657
47°	1.367	1.466	0.9325
48°	Ⓒ 1.346	1.494	0.9004
49°	1.325	1.524	0.8693
55°	1.221	1.743	0.7002
56°	1.206	Ⓐ 1.788	0.6745
57°	1.192	1.836	0.6494
58°	1.179	1.887	0.6249
65°	1.103	2.366	Ⓑ 0.4663
66°	1.095	2.459	0.4452
67°	1.086	2.559	0.4245
68°	1.079	2.669	0.4040
69°	1.071	2.690	0.3839

We find the number of degrees in the angle column, then look across in the same row under the appropriate column for the cosecant, the secant or the cotangent.

Ⓐ sec 56° = 1.788; Ⓑ cot 65° = 0.4663; Ⓒ csc 48° = 1.346

If you need to find the value of a reciprocal trigonometric ratio and you do not have a table of reciprocal values, you can take the reciprocal of the corresponding basic trigonometric value.

EXAMPLE:

$$\sec 35^\circ = \frac{1}{\cos 35^\circ} = \frac{1}{0.8192} = 1.221$$

This value is close to the value listed in the table on page 86 for sec 35°.

Self-Correcting Exercise #15

Use the table on page 86 to find the value of the following trigonometric ratios.

1. $\csc 12^\circ$ = ______ 2. $\cot 23^\circ$ = ______

3. $\sec 8^\circ$ = ______ 4. $\csc 57^\circ$ = ______

5. $\cot 51^\circ$ = ______ 6. $\sec 40^\circ$ = ______

7. $\csc 88^\circ$ = ______ 8. $\cot 44^\circ$ = ______

Check your answers with those on page 85 of this lesson.

EXERCISE: Finding Reciprocal Trigonometric Values

1. **When $\sin\theta = \frac{3}{5}$, find the remaining 5 trigonometric values. (See example on page 49)**

2. **Use the table on page 86 to find the value of:**

   (a) $\sec 23^\circ$ = ______ (b) $\cot 84^\circ$ = ______

   (c) $\csc 5^\circ$ = ______ (d) $\sec 62^\circ$ = ______

TOPIC THREE: STATISTICS

A. Introduction

Statistics is concerned with the collection of numerical facts called **data**, the organization and presentation of these facts, and finally, the interpretation of the data.

The subject of **statistics** has been applied to many different areas including insurance, medicine, advertising, electronics, television audiences, population growth, energy consumption, student enrollment, sports, and many others.

The following are some examples of the familiar use of statistics.

1. The Gallup Poll uses statistics to predict the outcome of elections.

2. Based on statistics, drivers under 25 years of age pay more for car insurance.

3. The Nielson television ratings help determine which T.V. programs should be discontinued.

4. Statistics show what brand of toothpaste is most effective in reducing tooth decay.

5. Schools are built or closed according to statistics.

To make statistics effective and meaningful, a statistician takes a sample which is a small group of individuals or objects selected to stand for a larger group called the **population.**

There are a number of problems that can arise in taking a sample. If the sample is too small, it may not be representative of the population. On the other hand a large sample is usually very costly.

The most important requirement of a valid sample is that it be a **random one.** This means that each individual in the population should have an equally likely chance of being selected. The **sampling** must be from a **target population** which means all possible individuals or objects are represented.

The facts collected by the pollsters are called **RAW DATA.** Raw data is the collection of numbers or data before it is arranged in some sort of order convenient for analysis. This raw data is organized and analyzed and inferences are made from it.

B. Frequency Distributions

Following is the raw data resulting from fifty rolls of a single die (singular of dice).

1 4 2 3 1   6 4 5 3 6   1 1 3 3 6   3 3 2 6 5   4 6 6 1 3

5 5 5 2 2   4 2 5 4 4   6 1 2 4 3   5 4 3 4 5   6 3 2 5 3

We can tabulate the data in a frequency distribution table in which we determine the number of times each value occurs.

Frequency Distribution Table*

Value	Tally Mark	Frequency
1	~~||||~~ |	6
2	~~||||~~ ||	7
3	~~||||~~ ~~||||~~ |	11
4	~~||||~~ ||||	9
5	~~||||~~ ||||	9
6	~~||||~~ |||	8

**A frequency distribution table is a grouping of values with a tally for each value. It tells us at a glance how frequently a particular value occurred. It is a useful way of organizing data to show any patterns the distribution may have.*

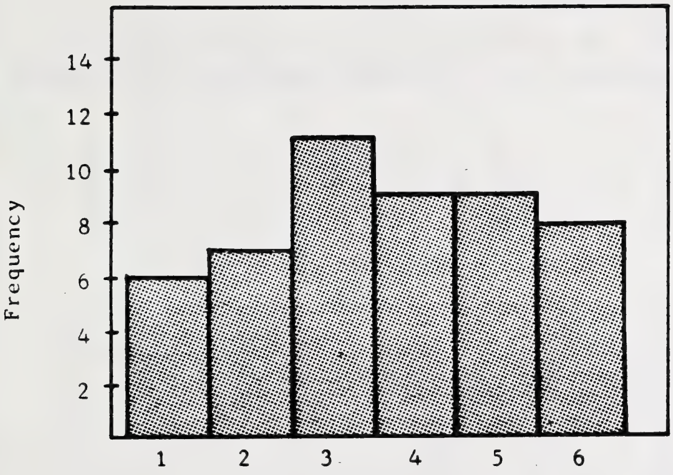
Formulating the data in a frequency table hides the order in which the value occurred. This information can be presented by means of graphs since they are a useful tool in picturing numerical information.

While some graphs give the total picture at first glance, others need a closer look. In general, a good graph will contain the following parts:

1. a title (this tells what relationship is being shown);
2. labelled axes;
3. a uniform scale on each axis (breaks in a scale are used where necessary).

A graph called a histogram is shown on the next page.

Histogram



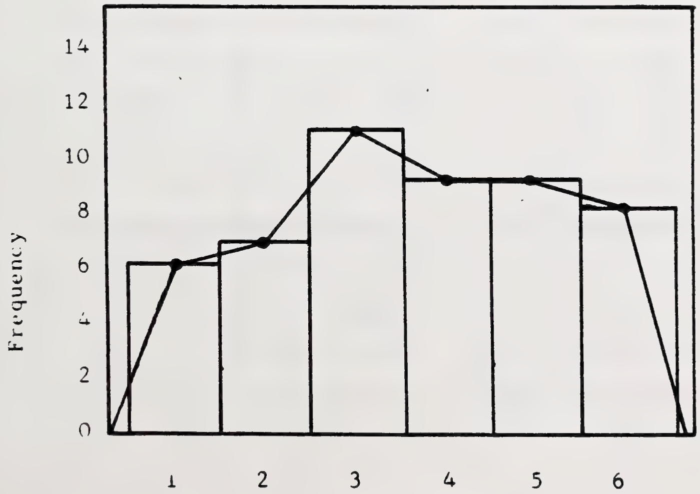
Observed Values

We can now read the frequencies from the vertical axis. When displaying a frequency distibution in a histogram, we are not really interested in the order, or in the individual values, but rather in general characteristics such as the **shape** of the distribution, most **frequent** scores and **range.*** If we join the **midpoints** of the tops of the rectangles of the histogram, the graph so formed (shown below) is called a **frequency polygon.** The distribution is extended one unit beyond the highest and lowest observed values so that the polygon is completed by drawing lines to these values on the baseline.

**Note:*

Please see page 69 for range.

Frequency Polygon



Observed Values

The **histogram** and the **frequency polygon** show the same **data.**

Exercise - Frequency Distribution

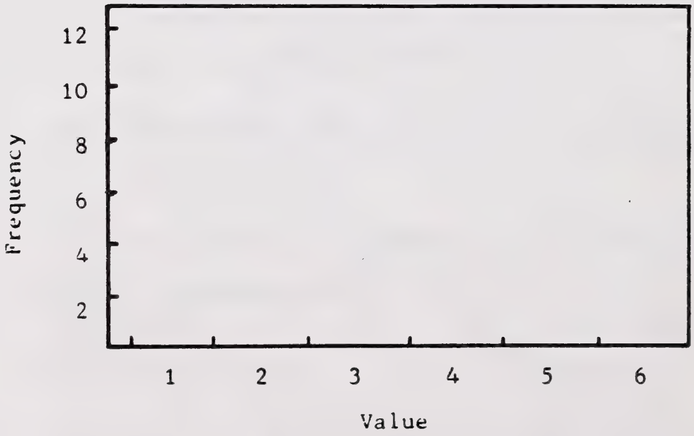
1. **Tabulate the following data to find the frequency, then construct a corresponding histogram and frequency polygon.**

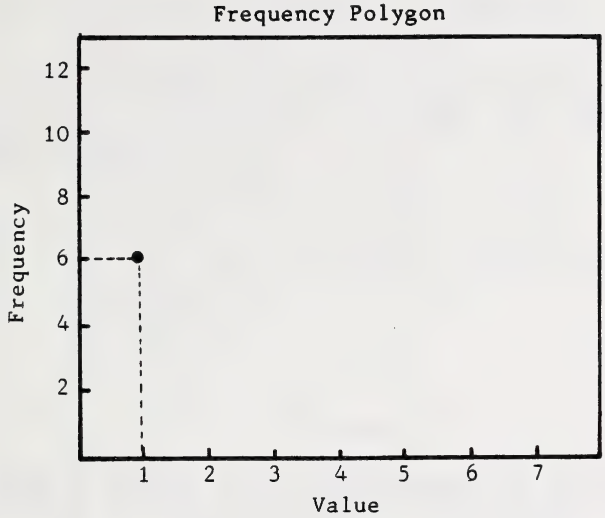
   4, 2, 4, 6, 1, 4, 2, 2, 3, 6, 1, 4, 4, 5, 2, 3, 4, 4, 5, 5, 1, 1, 6, 3, 4. 6, 3, 2, 3, 2, 4, 3, 3, 5, 2, 5, 6, 5, 2, 4, 6, 4, 6, 1, 2, 2, 5, 1, 3, 3.

Frequency Distibution Table

Value	Tally	Frequency
1	𝍸 |	6
2	𝍸 𝍸	10
3	𝍸 ||||	9
4	𝍸 𝍸 |	11
5	𝍸 ||	7
6	𝍸 ||	7

Histogram





2. Fifty applicants for employment at Acme Graphics were required to answer an aptitude test consisting of 50 questions.

   The raw scores were:

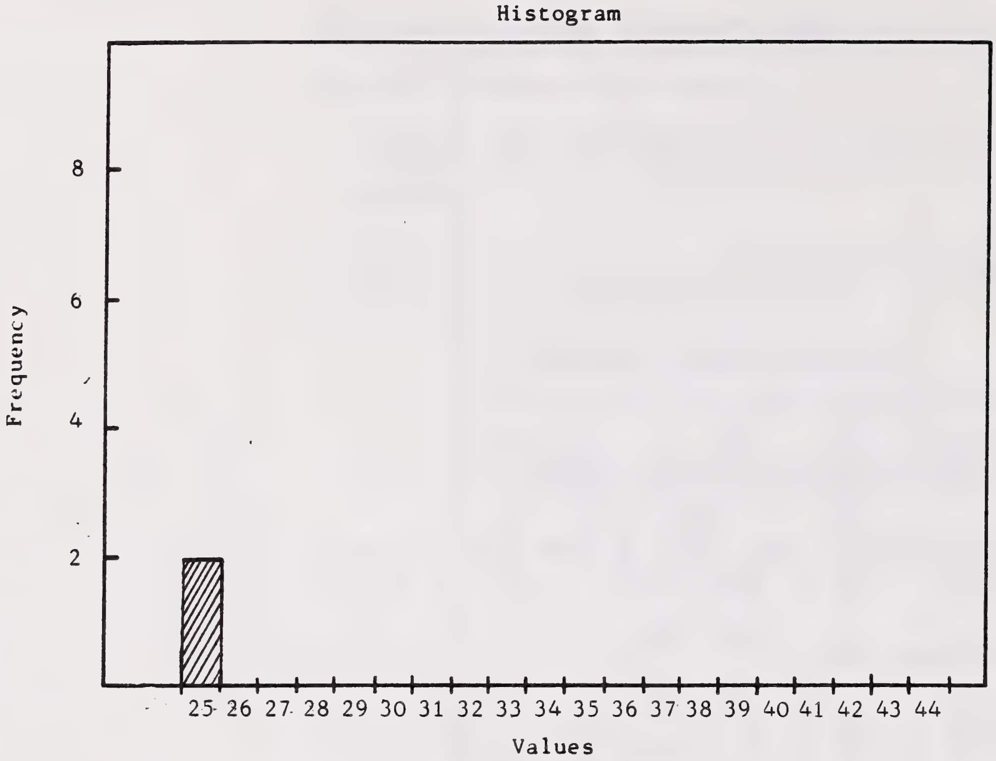
   25, 25, 26, 27, 28, 28, 29, 30, 30, 30, 30, 31, 31,
   31, 31, 32, 32, 32, 33, 33, 33, 34, 34, 34, 34, 34,
   35, 35, 35, 35, 35, 35, 35, 35, 36, 36, 36, 36, 37,
   37, 37, 37, 37, 38, 38, 38, 41, 41, 42, 43.

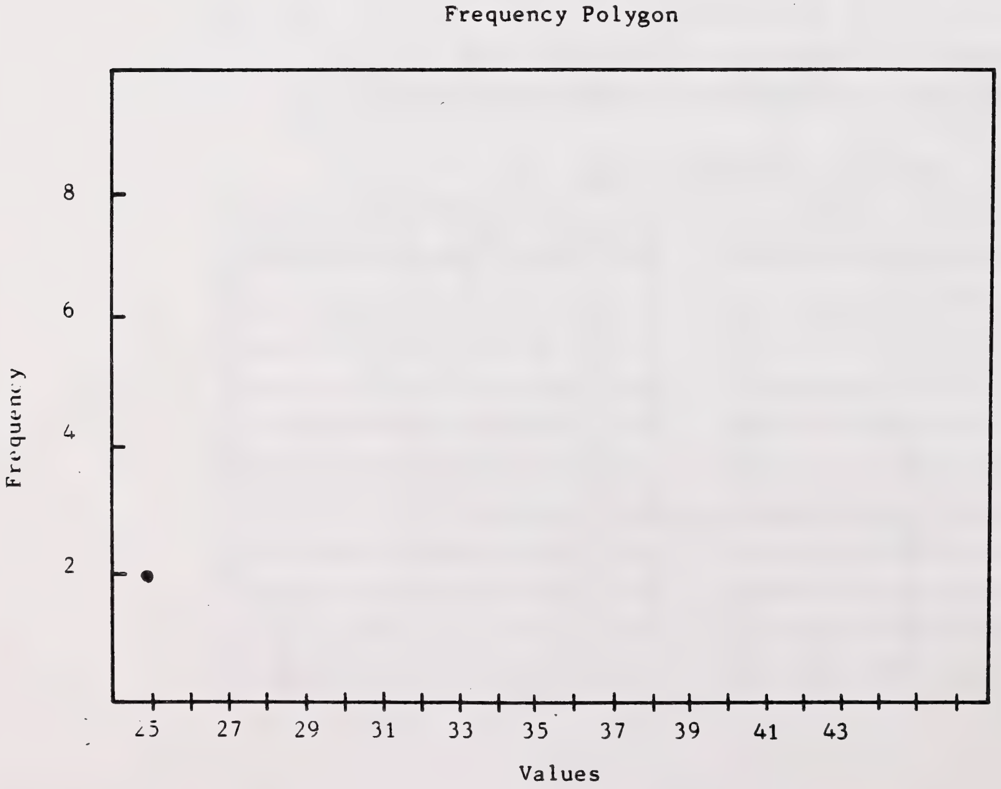
   (a) **Prepare a frequency distribution table and construct a corresponding histogram and frequency polygon.**

Frequency Distibution Table

Value	Tally	Frequency
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		

Value	Tally	Frequency
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		


Histogram
Frequency
8
6
4
2
25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44
Values


Frequency Polygon
Frequency
8
6
4
2
25
27
29
31
33
35
37
39
41
43
Values

C. Measures of Central Tendency

We often need to know where the middle of certain data is located. The measures that indicate the location of the middle or centre of the data are called the **MEASURES of CENTRAL TENDENCY.**

The three measures of central tendency are:

1. the mean (also called the arithmetic mean.)
2. the median
3. the mode

I. The Mean

The **MEAN** is usually called the average. The mean of a set of numbers is found by adding the numbers and then dividing the sum by the number of values added.

EXAMPLE: The sizes of the science classes in Laurier High School were as follows:

26, 31, 39, 32, 28, 33, 37, 27, 29, 31.

Find the mean size of a science class.

Solution

$$\text{Mean} = \frac{26 + 31 + 39 + 32 + 28 + 33 + 37 + 27 + 29 + 31}{10}$$

$$= \frac{313}{10} = 31.3$$

$$= 31 \text{ (rounded to the nearer whole number)}$$

The mean size of a science class is 31 students.

EXERCISE

1. A football player makes gains of 105 m, 86 m, 92 m, 120 m, and 75 m in five successive games. What is his mean gain per game?

2. A survey of wrist diameters, in centimetres, yielded the following:

5.8	6.1	5.5	6.3	5.9
5.2	6.4	5.7	6.1	

Calculate the mean wrist size.

II. The Median

The median is the **middle** measure of a set of data. When a set of numbers is arranged in order from smallest to largest, then the number in the middle is called the **MEDIAN.** With an odd number of values in the sample there is one number which is exactly in the middle. When we have an even number of values in the sample there are two numbers in the middle so to find the median we must find the mean of the two middle numbers.

EXAMPLES:

(a) **Find the median of the following test scores:**

62, 85, 71, 92, 63, 55, 70, 67, 51

Solution

First arrange the values in order from smallest to largest.

$$\underbrace{51 \quad 55 \quad 62 \quad 63}_{\textit{4 values}} \quad \underset{\substack{\uparrow \\ \textit{middle} \\ \textit{value}}}{67} \quad \underbrace{70 \quad 71 \quad 85 \quad 92}_{\textit{4 values}}$$

The median is 67.

(b) Find the median of the following test scores:

59, 83, 78, 69, 50, 75, 64, 57

Solution

Arrange the values in order.

$$50 \quad 57 \quad 59 \quad \underbrace{\boxed{64 \quad 69}}_{\textit{two middle values}} \quad 75 \quad 78 \quad 83$$

The two middle numbers are 64 and 69.

The median is $\frac{64 + 69}{2} = \frac{133}{2} = 66.5$

III. The Mode

The **MODE** of a set of numbers is the value that occurs the greatest number of times. There is **no mode** if each number occurs only once. If certain values occur the same number of times there will be more than one mode. If there are two modes the distribution is said to be **BI-MODAL**. The modes of distributions show up on bar graphs, histograms or frequency polygons as the peaks or highest bars.

This type of measure of central tendency is of great importance in retail merchandising because a retailer must know the sizes most in demand or the most popular products or brands in order to carry the correct stock.

EXAMPLES: Find the mode(s) of the following sets of numbers.

(a) 7, 8, 9. 7, 6, 4, 5, 7, 9, 10
(b) 8, 10, 6, 8, 7, 6, 19, 11, 8, 12, 6, 5
(c) 8, 6, 7, 4, 5, 3, 9, 10

Solution

(a) $4, \ 5, \ 6, \ \boxed{7, \ 7, \ 7,} \ 8, \ 9, \ 9, \ 10$

The mode is 7.

(b) $5, \ \boxed{6, \ 6, \ 6,} \ 7, \ \boxed{8, \ 8, \ 8,} \ 9, \ 10, \ 11, \ 12$

The modes are 6 and 8. This set is bi-modal.

(c) 3, 4, 5, 6, 7, 8, 9, 10

There is no mode.

Self-Correcting Exercise #16

1. The following donations were collected in one area in a charity drive.

$20, $8, $12, $25, $14, $25, $60, $70, $55, $25

Find: **(a) the mode**
(b) the median donation
(c) the mean donation

(a) Arrange the donations in order, smallest to largest.

____

The mode is ________.

(b) The median donation is ________.

(c) The mean donation is ________.
(Show work here.)

2. The property taxes paid by twelve home-owners in Blue Valley subdivision were as follows:

$720, $690, $910, $775, $905, $810
$1020, $775, $650, $825, $855, $742

Find the mode, median and mean.

(a) **Arrange the tax amounts in order. (Smallest to largest.)**

____

____

The mode is _______.

(b) The median is ________.

(c) The mean tax is ________.
(Show work here, round answer to nearer cent.)

3. **Tweeter and Woofer sell Hi-Fi equipment. Their sales for 5 weeks were as follows:**

Week	1	2	3	4	5
Tweeter	20	25	22	28	25
Woofer	18	28	12	29	28

Calculate the mode, median and mean for each salesperson.

(a) Mode for Tweeter:

Mode for Woofer:

(b) Median for Tweeter:

Median for Woofer:

(c) Mean for Tweeter:

Mean for Woofer:

Check your answers with those on page **84** of this lesson.

D. Selecting the Most Suitable Measure of Central Tendency

Each of the measures of central tendency (mode, median and mean) can be referred to as an average and each can be used to represent a set of data. However, often it is necessary to determine which of these special measures is the best representation for a particular set of data.

Sometimes an average is not very meaningful. There is a story about a man who drowned in a stream with an "average" depth of eleven inches. A hat check girl who received only two tips all evening, one of 25 cents and the other of ten dollars, would be unhappy if the management pointed out that her "average" tip was in excess of five dollars. These two examples show how the mean is not a good average when the set contains a few extreme cases of either high or low values.

The median is not distorted by a few extreme elements since it is a positional average. The median is also the most useful average when the addition of elements in a set is meaningless. Thus, if the measures under discussion are the intelligence quotients of the students in a class, the addition of the I.Q.'s is meaningless. When arranged in order, the median I.Q. would usually be a fair representation of the I.Q. of the average student.

The mode is apt to be distorted unless the set consists of many elements. If a student received marks of 25, 40, 80 and 80, the mode for the set would be 80 but it would certainly be a peculiar average. The mode is extremely useful in merchandising. It is only a rough measure of central tendency.

The mean and the median are used more frequently than the mode. The mean is used the most often since it can easily be found and doesn't require that the data be put in order. One disadvantage of the mean is that it is very much affected by extreme values (very high or very low values). In such cases, it is better to use the median as the representative value.

A mean or a median is reliable when a large number of items are included in a survey.

The mean, median and mode all locate the "middle" of the data but each best represents differents kinds of data.

The mean is useful for the following kind of data:

(i) averages in sports – batting averages, number of goals scored.
(ii) average snowfall, average rainfall, average temperature.
(iii) gas consumption of a car.

The median is useful for such data as:

(i) wages.
(ii) employee absentee days.
(iii) test scores.

The mode is useful for data that applies to:

(i) dress sizes.
(ii) hat sizes.
(iii) preferred kinds of cars.
(iv) most popular records.

EXERCISE – Measures of Central Tendency

Answers to those questions marked with a star, (*), are provided on page 84 of this lesson.

1. Kelly worked in her own small business along with her six employees, who earned the following yearly salaries: \$15 300, \$14 500, \$16 000, \$15 500, \$15 300, \$14 800. Kelly received \$38 750 as her yearly salary.

   Find the mean, median and mode and determine which is the best measure of central tendency for the given data.

   (a) Arrange the 7 salaries in order.

   ____________, ____________, ____________, ____________,

   ____________, ____________, ____________.

   The mode is ____________.

   (b) The median salary is ____________.

   (c) The mean salary is: (Show your work.)

Is this mean a good representation of the average yearly earnings of Kelly's employees? ____________________

Why? ____

____

The measure of central tendency which best indicates the average salary paid to Kelly's employees is the

____________.

*2. **Decide which measure (mean, median or mode) would best represent the following data.**

(a) the amount of precipitation your city receives in a year ____________

(b) the average mark on a science test ____________

(c) the most popular shirt size ____________

(d) the average number of points scored by soccer players in a game ____________

(e) the average wage of a factory worker ____________

(f) the most popular glove size ____________

(g) your bowling average ____________

(h) the average number of days absent by an employee ____________

3. The number of accidents at a particular intersection for nine consecutive months is shown below.

Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.
15	13	16	19	9	10	15	15	20

(a) **Find the mean, median and mode for the given data.**

Arrange the data in order.

____

The mode is ______.

The median is ______.

The mean is:

(b) The measure of central tendency that best represents the above data is the ____________ since there are no extreme values.

*4. Two sets of data, A and B, are as follows:

A: 12 14 15 18 20
B: 12 14 15 18 20 92

Find the mean and median for A and B.

Median for A is _________.

Mean for A is: (Show work to one decimal place)

Median for B is _________.

Mean for B is: (Show work to one decimal place)

The median is different for A and B because _________

____

The mean differs significantly for A and B because

____

Which measure of central tendency is the better representative for the B set of data? ______________

5. For each set of data find the mean and the median. Express answers to one decimal place.

(a) 6, 10, 8, 12, 10, 14, 9, 12, 13

Arrange the data in order smallest to largest.

____

The median is __________.

The mode is __________.

The mean is:

(b) 62, 58, 60, 52, 68, 73, 69, 64

Arrange the data in order from smallest to largest.

____

The median is __________.

The mode is __________.

The mean is:

(c) 6, 12, 9, 3, 0, 4, 8, 6, 5, 7, 9

Arrange the data in order from smallest to largest.

____

The median is __________.

The mode is __________.

The mean is:

E. Measures of Dispersion

The **mean, median** and **mode** are common measures of central tendency but, they do not tell everything we want to know about a set of data. Two sets of data could have the same **mean** or **median** and yet differ greatly in their **dispersion** or **spread.**

One measure of **dispersion** is the **range.**

The range of a set of numbers is found by subtracting the smallest number from the largest.

The **range** is very sensitive to extreme values but does not tell us anything about how the numbers vary. For this reason, we need another measure of the dispersion called the **standard deviation.** Standard deviation is represented by the symbol "σ" which is the **Greek letter sigma.** Standard deviation tells us something of the **spread** or **dispersion** of elements around the mean. In short, it is the amount by which an element **differs** from the **mean.**

The standard deviation of a relatively small set of numbers is determined by calculating the following:

(a) the mean of the numbers ($\overline{x}$);

(b) the difference between each number and the mean ($x_i - \overline{x}$);

(c) the squares of each of these differences $(x_i - \overline{x})^2$;

(d) the mean of the squares

$$\frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \ldots + (x_n - \overline{x})^2}{n}$$

(e) the square root of this mean

$$\sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}$$

The above steps result in the formula:

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}$$

OR

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$

$\sum$ ← *This symbol is the Greek capital letter sigma.*

It is referred to as a summation symbol.

EXAMPLE 1:

A baseball coach measured the heights of his players and found them to be 173 cm, 177 cm, 177 cm, 179 cm, 181 cm, 181 cm, 182 cm, 182 cm, 183 cm, and 185 cm. Find the range and the standard deviation of this set of data.

SOLUTION:

Range: 185 - 173 = 12

Standard deviation:

Step 1

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

$$\bar{x} = \frac{173+177+177+179+181+181+182+182+183+185}{10}$$

$$\bar{x} = \frac{1800}{10}$$

$$\bar{x} = 180$$

The mean is 180.

Step 2

Heights	Differences	Squares
x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
173	173 - 180 = -7	49
177	177 - 180 = -3	9
177	177 - 180 = -3	9
179	179 - 180 = -1	1
181	181 - 180 = 1	1
181	181 - 180 = 1	1
182	182 - 180 = 2	4
182	182 - 180 = 2	4
183	183 - 180 = 3	9
185	185 - 180 = 5	25

Step 3

The sum of the squares is:

$49 + 9 + 9 + 1 + 1 + 1 + 4 + 4 + 9 + 25 = 112$

Step 4

The mean of the squares is: $\frac{112}{10} = 11.2$

Step 5

$\sigma = \sqrt{11.2}$

$\sigma \doteq 3.3$

Therefore the standard deviation is approximately 3.3.

Exercise - Measures of Dispersion

1. **State the range of the following:**

(a) 3, 7, 15, 19, 24

Range = ______________________

(b) 13, 5, 36, 48, 9

Range = ______________________

(c) 54, 13, 25, 61, 47, 8

Range = ______________________

2. **Determine the standard deviation for the following sets of discrete data.**

(a) 6, 8, 8, 9, 9, 9, 10, 10, 12

The mean is: $\dfrac{6 + 8 + \underline{\hspace{6em}}}{9}$

$\bar{x}$ = ______________

Value	Differences	Squares
x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
6		
8		
8		
9		
9		
9		
10		
10		
12		

The mean of the squares is:

$= \dfrac{\underline{\hspace{10em}}}{9}$

= ______________

$\sigma = \sqrt{}$

σ = ____________ (rounded to one decimal place)

(b) 20, 24, 17, 15, 28, 25, 22, 20, 17, 12, 20

Mean is: $\frac{\rule{6cm}{0.4pt}}{11}$

= ____________

Value x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$

The sum of the squares is:

____

The means of the squares is = $\frac{\rule{2cm}{0.4pt}}{11}$ = ____________

$\sigma = \sqrt{}$

σ = ____________

Answers to Self-Correcting Exercises

Answers to questions, pages 3-4

1. 1 dm = 10 cm
2. 1 kg = 10 hg
3. 1 cL = 10 mL
4. 1 km = 10 hm
5. 1 dag = 10 g
6. 1 cm = 10 mm
7. 1 m = 1000 mm
8. 1 daL = 100 dL
9. 1 kg = 100 dag
10. 1 g = 100 cg
11. 1 L = 1000 mL
12. 1 g = 1000 mg
13. 1 cm = 0.1 dm
14. 1 dm = 0.1 m
15. 1 mg = 0.001 g
16. 1 mL = 0.1 cL
17. 1 cg = 0.01 g
18. 1 cm = 0.01 m
19. millimetre, kilometre, metre, or centimetre (any three of the above will do.)
20. kilolitre, litre and millilitre
21. kilogram, gram and milligram

Self-correcting Exercise #1, pages 5, 6

1.

Prefix	Symbol	Meaning in Words	Meaning in Numbers	Power of 10
mega	M	million	1 000 000	10^6
kilo	k	thousand	1 000	10^3
hecto	h	hundred	100	10^2
deca	da	ten	10	10^1
deci	d	tenth	0.1	10^{-1}
centi	c	hundredth	0.01	10^{-2}
milli	m	thousandth	0.001	10^{-3}
micro	µ	millionth	0.000 001	10^{-6}

2. (a) "kilometre" means 1000 m.
"Kilogram" means 1000 g.
"Kilolitre" means 1000 L.

(b) The prefix "centi" means hundredth or 0.01.
Thus "centimetre" means 0.01 m.
"Centigram" means 0.01 g.
"Centilitre" means 0.01 L.

(c) The prefix "milli" means thousandth or 0.001.
Thus "millimetre" means 0.001 m.
"Milligram" means 0.001 g.
"Millilitre" means 0.001 L.

3.

WORD NAME	**SI SYMBOL**
millimetre	mm
centimetre	cm
decimetre	dm
metre	m
decametre	dam
hectometre	hm
kilometre	km
kilogram	kg
gram	g
milligram	mg
kilolitre	kL
litre	L
millilitre	mL
centilitre	cL

4. (a) **micro, milli, centi, and deci**
   (b) **deca, hecto, kilo and mega**

5. (a) hectometre, [✓ decimetre]
   (b) [✓ millilitre], hectolitre
   (c) [✓ millimetre], metre
   (d) gram, [✓ decigram]
   (e) kilogram, [✓ decagram]
   (f) [✓ millimetre], centimetre
   (g) [✓ centimetre], decametre
   (h) decagram, [✓ decigram]
   (i) [✓ litre], kilolitre
   (j) [✓ litre], hectolitre

Self-correcting Exercise #2, pages 7, 8

1. (a) 10 cm, 1 dm
   (b) 10 mm, 100 mm
   (c) 0.5 cm, 2.0 cm, 4.2 cm, 6.7 cm, 9.1 cm
   (d) 10 mm, 25 mm, 54 mm, 77 mm, 99 mm
   (e) 4 cm, 2 cm, 5 cm
   (f) 35 mm, 26 mm, 43 mm

2.
9 m = 900 cm
3.6 m = 360 cm
0.4 m = 40 cm

5 m = 50 dm
2.2 m = 22 dm
0.7 m = 7 dm

3 m = 3000 mm
2.7 m = 2700 mm
4.75 m = 4750 mm

3.
50 cm = 0.5 m
200 cm = 2 m
1490 cm = 14.9 m

25 dm = 2.5 m
5.7 dm = 0.57 m
378 dm = 37.8 m

4500 mm = 4.5 m
910 mm = 0.91 m
75 mm = 0.075 m

Self-correcting Exercise #3, pages 9, 10

1. The prefix "kilo" means **thousand,** so to convert kilometres to metres you must **multiply** by the number **1000.**
2. 7 km = **7000** m; 15 km = **15 000** m; 0.5 km = **500 m**
3. The prefix "hecto" means **hundred,** so to convert hectometres to metres you must **multiply** by the number **100.**
4. 9 hm = **900** m; 25 hm = **2500** m; 0.7 hm = **70** m
5. The prefix "deca" means **ten,** so to convert decametres to metres you must **multiply** by the number **10.**
6. 3 dam = **30** m; 10.8 dam = **108** m; 64 dam = **640** m
7. To convert metres to decametres you must multiply by the decimal number **0.1.**
8. 19 m = **1.9** dam; 345 m = **34.5 dam;** 6 m = **0.6** dam
9. To convert metres to hectometres you must **multiply** by the decimal number **0.01.**
10. 136 m = **1.36** hm; 69 m = **0.69** hm; 4 m = **0.04** hm
11. To convert metres to kilometres you must **multiply** by the decimal number **0.001.**
12. 8000 m = 8.0 km; 270 m = **0.27** km;
    85 m = **0.085** km.

Answers to questions on the bottom of page 11

square millimetres	**mm^2**	square decametres	**dam^2**
square decimetres	**dm^2**	square hectometres	**hm^2**

Questions on page 12

1 cm = **10** mm, so 1 cm^2 = **10** mm × **10** mm
= **100** mm^2

1 dm = **10** cm, so 1 dm^2 = **10** cm × **10** cm
= **100** cm^2

Self-correcting Exercise #4, page 13

1. $1\text{ cm} = 10\text{ mm}$ $\quad$ $1\text{ dm} = 10\text{ cm}$
   $1\text{ cm}^2 = 100\text{ mm}^2$ $\quad$ $1\text{ dm}^2 = 100\text{ cm}^2$

2. $1\text{ hm}^2 = 10\ 000\text{ m}^2$ $\quad$ $50\ 000\text{ m}^2 = 5\text{ ha}$
   $3\text{ hm}^2 = 3\text{ ha}$ $\quad$ $22\ 000\text{ m}^2 = 2.2\text{ hm}^2$
   $7\text{ ha} = 70\ 000\text{ m}^2$ $\quad$ $9\text{ ha} = 90\ 000\text{ m}^2$

3. (a) $\text{Area} = \text{length} \times \text{width} = 25\text{ mm} \times 20\text{ mm} = 500\text{ mm}^2$

   (b) $\text{Area} = \text{length} \times \text{width} = 4.1\text{ cm} \times 3.2\text{ cm} = 13.12\text{ cm}^2$

4. (a) $\text{Length} = \dfrac{\text{Area}}{\text{Width}} = \dfrac{544\text{ cm}^2}{16\text{ cm}} = 34\text{ cm}$

   (b) $\text{Length} = \dfrac{\text{Area}}{\text{width}} = \dfrac{1575\text{ mm}^2}{35\text{ mm}} = 45\text{ mm}$

Answers to questions on the top of page 15

cubic decimetre	**dm^3**	cubic hectometre	**hm^3**
cubic kilometre	**km^3**	cubic decametre	**dam^3**

Self-correcting Exercise #5, page 15

1. The smallest cube in the diagram on page 14 has an edge of 1 mm and its volume is $1\text{ mm} \times 1\text{ mm} \times 1\text{ mm} = 1\text{ mm}^3$.

2. The centimetre cube in the diagram has an edge of 1 cm or 10 mm, so its volume is 1 cm^3 or $10\text{ mm} \times 10\text{ mm} \times 10\text{ cm} = 1000\text{ mm}^3$.

3. The edge of the decimetre cube measures 1 dm or 10 cm, so its volume is 1 dm^3 or $10\text{ cm} \times 10\text{ cm} \times 10\text{ cm} = 1000\text{ cm}^3$.

Self-correcting Exercise #6, page 16

1. kilo means thousand $\quad$ 1 kL = 1000 L
   hecto means **hundred** $\quad$ 1 hL = 100 L
   deca means **ten** $\quad$ 1 daL = 10 L
   deci means **tenth** $\quad$ 1 dL = **0.1** L
   centi means **hundredth** $\quad$ 1 cL = **0.01** L
   milli means **thousandth** $\quad$ 1 mL = 0.001 L

2. 1 kL = 10 hL $\quad$ 1 dL = **0.1** L
   1 hL = 10 daL $\quad$ 1 daL = **0.1** hL
   1 cL = **0.1** dL $\quad$ 1 mL = **0.01** dL
   1 mL = **0.1** cL $\quad$ 1 hL = **0.1** kL
   1 kL = **100** daL $\quad$ 1 daL = **0.01** kL

Self-Correcting Exercise #7, page 17.

1. 1 mL = 1 cm^3
16 mL = 16 cm^3
2.5 cm^3 = 2.5 mL
1 dm^3 = 1L
18 dm^3 = 18 L
320 L = 320 dm^3

2. Volume = 25 cm × 12 cm × 10 cm
= 3000 cm^3
= 3000 mL
= 3 L

The capacity of the box is 3000 mL or 3 L.

Self-Correcting Exercise #8, pages 18, 19.

1. One kilogram (1 kg) = 1000 g
One hectogram (1 hg) = 100 g
One decagram (1 dag) = 10 g
One gram (1 g) = 1 g
One decigram (1 dg) = 0.1 g
One centigram (1 cg) = 0.01 g
One milligram (1 mg) = 0.001 gm

2. 1 kg = 1000 g
7 kg = 7000 g
12.2 kg = 12 200 g
1 g = 0.001 kg
3 g = 0.003 kg
52.5 g = 0.0525 kg

3. 1 g = 1000 mg
5 g = 5000 mg
43 g = 43 000 mg
1 mg = 0.001 g
13 mg = 0.013 g
64.5 mg = 0.0645 g

Self-Correcting Exercise #9, page 20.

1. 1 cm^3 of water is equivalent to 1 mL and weighs 1 g.

2. 95 cm^3 of water are equivalent to 95 mL and weigh 95 g.

3. 1 dm^3 of water is equal to 1 L and weighs 1 kg.

4. One cubic metre has a capacity of 1 kL or 1000 L.

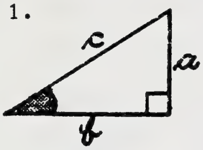
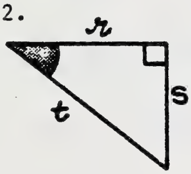
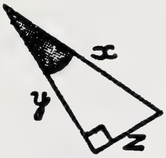
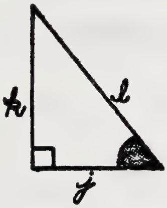
5. The weight of 1 kL of water is 1 t.

6. 10.5 m^3 of water is equivalent to 10.5 kL and weigh 10.5 t.

7. Volume = 26 cm × 26 cm × 26 cm
= 17 576 cm^3
= 17 576 mL

The weight of water in grams is 17 576 g and in kilograms is 17.576 kg.

Self-Correcting Exercise #10, pages 26 and 27.

Triangle	Hypotenuse	Side Opposite	Side Adjacent
1. 	c	a	b
2. 	t	s	r
3. 	x	z	y
4. 	l	k	j

Self-Correcting Exercise #11, page 30.

1.
(a) $\sin 20^\circ = 0.3420$
(b) $\tan 86^\circ = 14.30$
(c) $\cos 78^\circ = 0.2079$
(d) $\sin 90^\circ = 1.000$
(e) $\cos 0^\circ = 1.000$
(f) $\tan 0^\circ = 0.000$

2.
(a) $\sin \theta = 0.5299$
$\theta = 32^\circ$

(b) $\tan \theta = 1.963$
$\theta = 63^\circ$

(c) $\sin \theta = 0.9998$
$\theta = 89^\circ$

(d) $\cos \theta = 0.3256$
$\theta = 71^\circ$

Self-Correcting Exercise #12, page 31.

1. $\cos\theta = 0.9850$

   $\theta = 10^\circ$

   Explanation

   $\cos 9^\circ = .9877 \longleftarrow$ *.9850 lies between.*

   $\cos 10^\circ = .9848$

   .9850 is clearly closer to .9848 so that the θ value we want is 10°.

2. $\cos\theta = 0.5055$

   $\theta = 60^\circ$

   Explanation

   $\cos 59^\circ = .5150 \longleftarrow$ *.5055 lies between*

   $\cos 60^\circ = .5000$

   Subtract to determine which cosine value is closer to .5055

$$\begin{array}{rcr} .5150 & \textit{given} \longrightarrow & .5055 \\ \underline{-.5055} \longleftarrow & \textit{value} & \underline{-.5000} \\ .0095 & & .0055 \end{array}$$

   Since this is the smaller difference .5000 is the closer value so the θ value we want is 60°.

3. $\sin\theta = 0.7500$

   $\theta = 49^\circ$

   Explanation

   $\sin 48^\circ = .7431 \longleftarrow$ *.7500 lies between.*

   $\sin 49^\circ = .7547$

   Subtract to determine which sine value is closer to .7500.

$$\begin{array}{rcr} .7500 \longleftarrow & \textit{given} & .7547 \\ \underline{-.7431} & \textit{value} \longrightarrow & \underline{-.7500} \\ .0069 & & .0047 \end{array}$$

   Since this is the smaller difference, .7547 is the closer value so the θ value we want is 49°.

4. $\sin\theta = 0.9999$

   $\theta = 90^\circ$

   Explanation

   $\sin 89^\circ = .9998 \longleftarrow$ *.9999 lies between.*

   $\sin 90^\circ = 1.000$

   .9999 lies exactly half-way between .9998 and 1.000. Therefore, we must round to the higher degree and use the θ value of 90°.

The calculator value, to the nearest degree may be 89°.

5. $\tan\theta = 0.0500$

$\theta = 3^\circ$

Explanation:

$\tan 2^\circ = .0349$ ← *.0500 lies between*

$\tan 3^\circ = .0524$

Subtracting, we get:

$$\begin{array}{r} .0500 \\ -.0349 \\ \hline .0151 \end{array} \quad \xleftarrow{\text{given}} \;\; \xrightarrow{\text{value}} \quad \begin{array}{r} .0524 \\ -.0500 \\ \hline .0024^{*} \end{array}$$

smaller difference

.0524 is the closer value and the θ value we want is 30°.

6. $\cos\theta = 0.2000$

$\theta = 78^\circ$

Explanation:

$\cos 78^\circ = .2079$ ← *.2000 lies between*

$\cos 79^\circ = .1908$

Subtracting, we get:

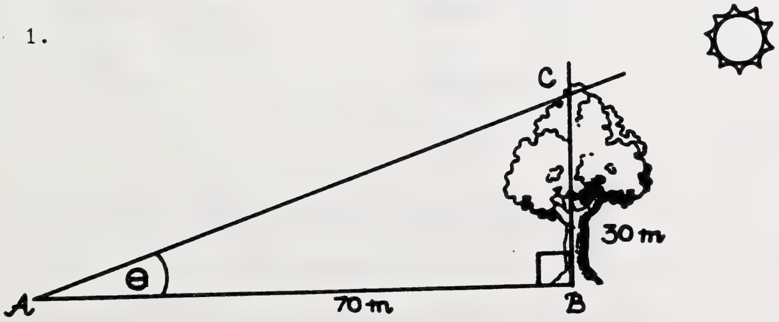
$$\begin{array}{r} .2079 \\ -.2000 \\ \hline {}^{*}.0079 \end{array} \quad \xleftarrow{\text{value}} \;\; \xrightarrow{\text{given}} \quad \begin{array}{r} .2000 \\ -.1908 \\ \hline .0092 \end{array}$$

smaller difference

.2079 is the closer value and the θ value we want is 78°.

Self-Correcting Exercise #13, pages 44 and 45.

1.

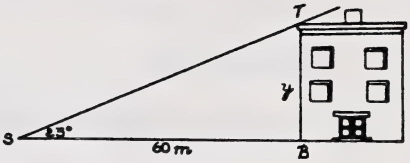


$$\tan\theta = \frac{30}{70} = 0.4286$$

$$= 23.2^\circ$$

The angle of elevation of the sun is 23.2°.

2. $$\tan 23° = \frac{y}{60}$$



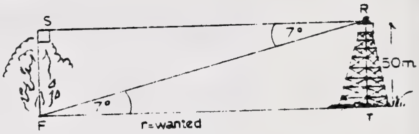
$$0.4245 = \frac{y}{60}$$

$$y = 60 \times 0.4245$$

$$y = 25.4700$$

The building is 25.5 m tall.

3. $$\tan 7° = \frac{50}{r}$$



$$0.1228 = \frac{50}{r}$$

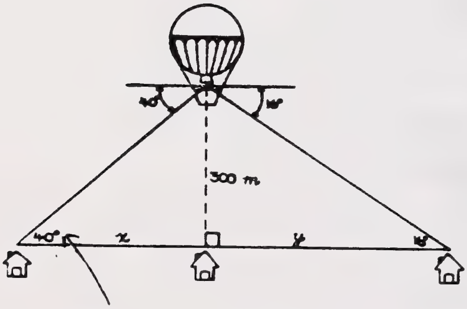
$$0.1228 \times r = 50$$

$$r = \frac{50}{0.1228}$$

$$r = 407.17$$

The fire was 407.2 m away.

4. We need to find the distances x and y separately. Then take the sum to find the distance between the first and last farmhouse.



$$\tan 40° = \frac{300}{x} \qquad \tan 16° = \frac{300}{y}$$

$$x = \frac{300}{\tan 40°} \qquad y = \frac{300}{\tan 16°}$$

$$x = \frac{300}{0.8391} \qquad y = \frac{300}{0.2867}$$

$$x = 357.5 \qquad y = 1046.4$$

$$\text{Total Distance} = 357.5 + 1046.4$$
$$= 1403.9 \text{ m}$$

The first and last farmhouses are 1403.9 m apart.

Self-Correcting Exercise #14, page 50.

1. (a) $\sin\theta = \frac{5}{\sqrt{41}}$ $\qquad \csc\theta = \frac{\sqrt{41}}{5}$

$\cos\theta = \frac{4}{\sqrt{41}}$ $\qquad \sec\theta = \frac{\sqrt{41}}{4}$

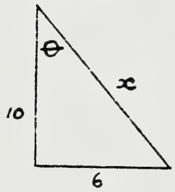
$\tan\theta = \frac{5}{4}$ $\qquad \cot\theta = \frac{4}{5}$

(b) $\sin\theta = \frac{7}{\sqrt{74}}$ $\qquad \csc\theta = \frac{\sqrt{74}}{7}$

$\cos\theta = \frac{5}{\sqrt{74}}$ $\qquad \sec\theta = \frac{\sqrt{74}}{5}$

$\tan\theta = \frac{7}{5}$ $\qquad \cot\theta = \frac{5}{7}$

2.



First, use the Pythagorean Theorem to find x.

$$x^2 = 10^2 + 6^2$$

$$x^2 = 100 + 36$$

$$x^2 = 136$$

$$x = \sqrt{136}$$

$\sin\theta = \frac{6}{\sqrt{136}}$ $\qquad \csc\theta = \frac{\sqrt{136}}{6}$

$\cos\theta = \frac{10}{\sqrt{136}}$ $\qquad \sec\theta = \frac{\sqrt{136}}{10}$

$\cot\theta = \frac{10}{6}$

Self-Correcting Exercise #15, page 52.

1. csc 12° = 4.810 $\qquad$ 2. cot 23° = 2.3559

3. sec 8° = 1.010 $\qquad$ 4. csc 57° = 1.192

5. cot 51° = 0.8098 $\qquad$ 6. sec 40° = 1.305

7. csc 88° = 1.001 $\qquad$ 8. cot 44° = 1.0355

Self-Correcting Exercise #16, pages 62 and 63.

1. (a) 8, 12, 14, 20, 25, 25, 25, 55, 60, 70

The mode is \$25.

(b) The median donation is $\frac{25 + 25}{2}$ = \$25.

(c) The mean donation is $\frac{314}{10}$ = \$31.4.

2. (a) \$650, \$690, \$720, \$742, \$775, \$775, \$810, \$825, \$855, \$905, \$910, \$1020

The mode is \$775.

(b) The median tax is $\frac{775 + 810}{2}$ = \$792.50.

(c) The mean tax is \$806.42.

3. (a) Mode for Tweeter is 25

Mode for Woofer is 28.

(b) Median for Tweeter is 25.

Mode for Woofer is 28.

(c) Mean for Tweeter is 24.

Mean for Woofer is 23.

Answers to #2 on page 66.

(a) mean *(b) median (c) mode (d) mean

(e) median (f) mode (g) mean (h) median

*This measure would be easy to use if the test scores are arranged in order. The mean can also be a measure of central tendency of test scores.

Answers to #4 on page 67.

Median for A is 15. Mean for A is 15.8.

Median for B is 16.5. Mean for B is 28.5.

The median is different for A and B because B has one more element.

The mean is different because B contains 92 which is much larger than the other numbers.

The median is a better measure of central tendency for B.

Sines, Cosines and Tangents 0° - 90°

ANGLE deg	sin	cos	tan
0	0.0000	1.0000	0.0000
1	0.0175	0.9998	0.0175
2	0.0349	0.9994	0.0349
3	0.0523	0.9986	0.0524
4	0.0698	0.9976	0.0699
5	0.0872	0.9962	0.0875
6	0.1045	0.9945	0.1051
7	0.1219	0.9925	0.1228
8	0.1392	0.9903	0.1405
9	0.1564	0.9877	0.1584
10	0.1736	0.9848	0.1763
11	0.1908	0.9816	0.1944
12	0.2079	0.9781	0.2126
13	0.2250	0.9744	0.2309
14	0.2419	0.9703	0.2493
15	0.2588	0.9659	0.2679
16	0.2756	0.9613	0.2867
17	0.2924	0.9563	0.3057
18	0.3090	0.9511	0.3249
19	0.3256	0.9455	0.3443
20	0.3420	0.9397	0.3640
21	0.3584	0.9336	0.3839
22	0.3746	0.9272	0.4040
23	0.3907	0.9205	0.4245
24	0.4067	0.9135	0.4452
25	0.4226	0.9063	0.4663
26	0.4384	0.8988	0.4877
27	0.4540	0.8910	0.5095
28	0.4695	0.8829	0.5317
29	0.4848	0.8746	0.5543
30	0.5000	0.8660	0.5774
31	0.5150	0.8572	0.6009
32	0.5299	0.8480	0.6249
33	0.5446	0.8387	0.6494
34	0.5592	0.8290	0.6745
35	0.5736	0.8192	0.7002
36	0.5878	0.8090	0.7265
37	0.6018	0.7986	0.7536
38	0.6157	0.7880	0.7813
39	0.6293	0.7771	0.8098
40	0.6428	0.7660	0.8391
41	0.6561	0.7547	0.8693
42	0.6691	0.7431	0.9004
43	0.6820	0.7314	0.9325
44	0.6947	0.7193	0.9657
45	0.7071	0.7071	1.0000

ANGLE deg	sin	cos	tan
45	0.7071	0.7071	1.0000
46	0.7193	0.6947	1.0355
47	0.7314	0.6820	1.0724
48	0.7431	0.6691	1.1106
49	0.7547	0.6561	1.1504
50	0.7660	0.6428	1.1918
51	0.7771	0.6293	1.2349
52	0.7880	0.6157	1.2799
53	0.7986	0.6018	1.3270
54	0.8090	0.5878	1.3764
55	0.8192	0.5736	1.4281
56	0.8290	0.5592	1.4826
57	0.8387	0.5446	1.5399
58	0.8480	0.5299	1.6003
59	0.8572	0.5150	1.6643
60	0.8660	0.5000	1.7321
61	0.8746	0.4848	1.8040
62	0.8829	0.4695	1.8807
63	0.8910	0.4540	1.9626
64	0.8988	0.4384	2.0503
65	0.9063	0.4226	2.1445
66	0.9135	0.4067	2.2460
67	0.9205	0.3907	2.3559
68	0.9272	0.3746	2.4751
69	0.9336	0.3584	2.6051
70	0.9397	0.3420	2.7475
71	0.9455	0.3256	2.9042
72	0.9511	0.3090	3.0777
73	0.9563	0.2924	3.2709
74	0.9613	0.2756	3.4874
75	0.9659	0.2588	3.7321
76	0.9703	0.2419	4.0108
77	0.9744	0.2250	4.3315
78	0.9781	0.2079	4.7046
79	0.9816	0.1908	5.1446
80	0.9848	0.1736	5.6713
81	0.9877	0.1564	6.3138
82	0.9903	0.1392	7.1154
83	0.9925	0.1219	8.1443
84	0.9945	0.1045	9.5144
85	0.9962	0.0872	11.4301
86	0.9976	0.0698	14.3007
87	0.9986	0.0523	19.0811
88	0.9994	0.0349	28.6363
89	0.9998	0.0175	57.2900
90	1.0000	0.0000	

undefined1.0000 undefined0.0000

Cosecants, Secants and Cotangents 0° – 90°

ANGLE deg	csc	sec	cot
0	undefined	1.0000	undefined
1	57.299	1.0002	57.2900
2	28.654	1.0006	28.6363
3	19.107	1.0014	19.0811
4	14.336	1.0024	14.3007
5	11.474	1.0038	11.4301
6	9.5668	1.0055	9.5144
7	8.2055	1.0075	8.1443
8	7.1853	1.0098	7.1154
9	6.3925	1.0125	6.3138
10	5.7588	1.0154	5.6713
11	5.2408	1.0187	5.1446
12	4.8097	1.0223	4.7046
13	4.4454	1.0263	4.3315
14	4.1336	1.0306	4.0108
15	3.8637	1.0353	3.7321
16	3.6280	1.0403	3.4874
17	3.4208	1.0457	3.2709
18	3.2361	1.0515	3.0777
19	3.0716	1.0576	2.9042
20	2.9238	1.0642	2.7475
21	2.7904	1.0711	2.6051
22	2.6695	1.0785	2.4751
23	2.5593	1.0864	2.3559
24	2.4586	1.0946	2.2460
25	2.3662	1.1034	2.1445
26	2.2812	1.1126	2.0503
27	2.2027	1.1223	1.9626
28	2.1301	1.1326	1.8807
29	2.0627	1.1434	1.8040
30	2.0000	1.1547	1.7321
31	1.9416	1.1667	1.6643
32	1.8871	1.1792	1.6003
33	1.8361	1.1924	1.5399
34	1.7883	1.2062	1.4826
35	1.7435	1.2208	1.4281
36	1.7013	1.2361	1.3764
37	1.6616	1.2521	1.3270
38	1.6243	1.2690	1.2799
39	1.5890	1.2868	1.2349
40	1.5557	1.3054	1.1918
41	1.5243	1.3250	1.1504
42	1.4945	1.3456	1.1106
43	1.4663	1.3673	1.0724
44	1.4396	1.3902	1.0355
45	1.4142	1.4142	1.0000

ANGLE deg	csc	sec	cot
45	1.4142	1.4142	1.0000
46	1.3901	1.4396	0.9657
47	1.3673	1.4663	0.9325
48	1.3456	1.4945	0.9004
49	1.3250	1.5243	0.8693
50	1.3054	1.5557	0.8391
51	1.2868	1.5890	0.8098
52	1.2690	1.6243	0.7813
53	1.2521	1.6616	0.7536
54	1.2361	1.7013	0.7265
55	1.2208	1.7435	0.7002
56	1.2062	1.7883	0.6745
57	1.1924	1.8361	0.6494
58	1.1792	1.8871	0.6249
59	1.1666	1.9416	0.6009
60	1.1547	2.0000	0.5774
61	1.1434	2.0627	0.5543
62	1.1326	2.1301	0.5317
63	1.1223	2.2027	0.5095
64	1.1126	2.2812	0.4877
65	1.1034	2.3662	0.4663
66	1.0946	2.4586	0.4452
67	1.0864	2.5593	0.4245
68	1.0785	2.6695	0.4040
69	1.0712	2.7904	0.3839
70	1.0642	2.9238	0.3640
71	1.0576	3.0716	0.3443
72	1.0515	3.2361	0.3249
73	1.0457	3.4203	0.3057
74	1.0403	3.6280	0.2867
75	1.0353	3.8637	0.2679
76	1.0306	4.1336	0.2493
77	1.0263	4.4454	0.2309
78	1.0223	4.8097	0.2126
79	1.0187	5.2408	0.1944
80	1.0154	5.7588	0.1763
81	1.0125	6.3925	0.1584
82	1.0098	7.1853	0.1405
83	1.0075	8.2055	0.1228
84	1.0055	9.5668	0.1051
85	1.0038	11.474	0.0875
86	1.0024	14.336	0.0699
87	1.0014	19.107	0.0524
88	1.0006	28.654	0.0349
89	1.0002	57.299	0.0175
90			

undefined1.0000 undefined0.0000

END OF LESSON K

Mathematics Upgading

Answer Key for Lessons A to K

ANSWER KEY

LESSON A

Page 2

Write in symbols.

3 is an element of set B. **$3 \in B$**

8 is not an element of set B. **$8 \notin B$**

Page 5

2. **B = {all even natural numbers}**
3. **C = {all multiples of 6}**
4. **D = {all months of the year beginning with the letter J}**

Page 6

(i) **$x > 7$**

(ii) **$x \leq 17$**

Use set-builder notation to specify the set of all natural numbers less than 15.

$\{x \mid x < 15, x \in N\}$

Now tabulate this set. **$\{1,2,3,\ldots,14\}$**

Page 7

(i) $1 < x < 9$

(ii) $6 < x \leq 50$

(iii) $4 \leq x \leq 40$

(iv) $3 < x < 25$

2. $\{x \mid 5 \leq x \leq 10, x \in N\}$

less than or equal to 10

3. $\{x \mid 2 \leq x < 7, x \in N\}$

"The set of all x such that x is greater than or equal to 2 and less than 7 and x is a natural number."

Page 9

Tabulate another finite set. **{2,4,6,...,24}** or any other finite set.

How many elements does this set have? ______ (This can be any number depending on your choice of set.)

Tabulate another infinite set. **{1,3,5,...}** or any other infinite set.

Specify another empty set. $\{x|x < 1, x \in N\}$ (This can be any other empty set.)

Page 11

Is set B a subset of set C? **No.** Why? **Set B contains the element 1 which does not belong to set C.**

Is set C a subset of set B? **Yes.** Why? **Every element of C is also an element of set B.**

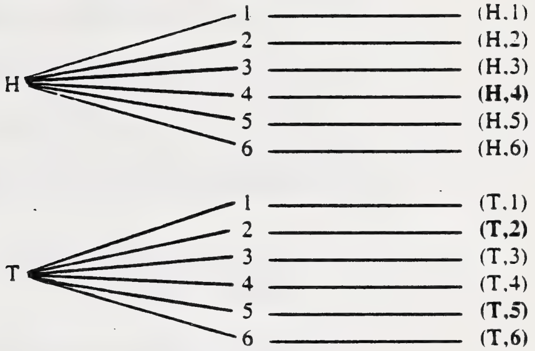
Is set C a subset of set A? **No.** Why? **Set C contains the element 4 which is not an element of set A.**

Set	Number of Elements in Set	Subsets	Number of Subsets
{ }	0	{ }	1
{a}	1	{ }, {a}	2
{a,b}	**2**	**{ }, {a}, {b}, {a,b}**	**4**
{a,b,c}	**3**	**{ }, {a}, {b}, {c}, {a,b}, {b,c}, {a,c}, {a,b,c}**	**8**
{a,b,c,d}	**4**	{ }, {a}, {b}, {c}, {d}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d}, {a,b,c}, **{a,c,d}, {a,b,d}, {b,c,d}, {a,b,c,d}**	16

A set with 2 elements has $2^2 = \mathbf{4}$ subsets.

A set with 3 elements has $\mathbf{2^3} = \mathbf{8}$ subsets.

A set with 4 elements has $\mathbf{2^4} = \mathbf{16}$ subsets.



What does the ordered pair (T,2) tell you?

A tail was obtained on the coin and a 2 was rolled on the die.

Which ordered pairs correspond...**(H,4), (H,5), (H,6)**

Pages 21-27

1. (a) is an element of (b) tabulating
   (c) natural (d) set-builder, "such that"
   (e) finite (f) null, empty, Ø, { }
   (g) subset (h) union
   (i) ordered (j) union, intersection, and Cartesian Product

2. (b) B = {February}, finite (c) C = {7,14,21,...}, infinite
   (d) D = Ø or { }, null (e) E = {Sun,Mon,Tues,Wed,Thurs,Fri,Sat}, finite
   (f) F = {5,6,7,...}, infinite (g) G = {m,i,s,l,e}, finite
   (h) H = Ø or { }, null (i) I = {5,10,15,...,135}, finite
   (j) J = {16,25,36,49,64,81}, finite (k) K = {1,3,5,...}, infinite

3. (a) The letters "e" and "t" have been listed more than once.
   (b) 9 should not be included because 9 is not less than 9.
   (c) The commas have been omitted.
   (d) The number 100 should not be included because it has three digits.

4. (b) $\{x \mid x > 12,\ x \in N\}$
$\{13, 14, 15, \ldots\}$

(c) $\{x \mid 9 < x \leq 36,\ x \in N\}$
$\{10, 11, 12, \ldots, 36\}$

(d) $\{x \mid 5 < x < 8,\ x \in N\}$
$\{6, 7\}$

(e) $\{x \mid x \leq 100,\ x \in N\}$
$\{1, 2, 3, \ldots, 100\}$

(f) $\{x \mid 3 \leq x < 42,\ x \in N\}$
$\{3, 4, 5, \ldots, 41\}$

5. (b) $\begin{cases} A \cup B = \{\mathbf{1,2,3,4,9,25,36}\} \\ A \cap B = \{\mathbf{1,4}\} \end{cases}$

(c) $\begin{cases} A \cup B = \{\textbf{black, green, red, yellow}\} \\ A \cap B = \{\ \} \end{cases}$

(d) $\begin{cases} A \cup B = \{\mathbf{3,4,5,6}\} \\ A \cap B = \{\mathbf{4,5}\} \end{cases}$

(e) $\begin{cases} A \cup B = \{\mathbf{1,2,3}\} \\ A \cap B = \{\ \} \end{cases}$

(f) $\begin{cases} A \cup B = \{\mathbf{15,20,25,26,27,51,60,75,100}\} \\ A \cap B = \{\mathbf{15,60}\} \end{cases}$

(g) $\begin{cases} A \cup B = \{\mathbf{1,2,3,\ldots}\} \\ A \cap B = \{\ \} \end{cases}$

(h) $\begin{cases} A \cup B = \{\mathbf{1,2,3,\ldots}\} \\ A \cap B = \{\mathbf{2,4,6,\ldots}\} \end{cases}$

(i) $\begin{cases} A \cup B = \{\mathbf{2,3,5}\} \\ A \cap B = \{\mathbf{2,3,5}\} \end{cases}$

(j) $\begin{cases} A \cup B \cup C = \{\mathbf{1,2,3,4,5,7}\} \\ A \cap B \cap C = \{\mathbf{3}\} \end{cases}$

(k) $\begin{cases} A \cup B \cup C = \{\mathbf{a,b,c,d,e}\} \\ A \cap B \cap C = \{\mathbf{c,d}\} \end{cases}$

6. $\{\ \}$, $\{\mathbf{7}\}$, $\{\mathbf{8}\}$, $\{\mathbf{9}\}$, $\{\mathbf{7,8}\}$, $\{\mathbf{7,9}\}$, $\{\mathbf{8,9}\}$, $\{\mathbf{7,8,9}\}$

7. (a) $\{\mathbf{10,40,50,60}\}$, $\{\mathbf{30,40,50,60}\}$, $\{\mathbf{20,30,50,60}\}$ (or any other four elements from set S)

(b) $\{\mathbf{10,20,30,40,50}\}$, $\{\mathbf{10,20,40,50,60}\}$, $\{\mathbf{20,30,40,50,60}\}$ (or any other five elements from set S)

(c) $\{10\}$, $\{20\}$, $\{30\}$, $\{40\}$, $\{50\}$, $\{60\}$

8. (a) **{(3,2),(3,4),(3,6),(4,2),(4,4),(4,6)}**

   (b) **{(2,3),(2,4),(4,3),(4,4),(6,3),(6,4)}**

   (c) **{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)}**

   (d) **{(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5)}**

   (e) **{4}**

   (f) **{1,2,3,4,5,6}**

9. (a) true

   (b) false

   (c) false

   (d) true

   (e) false

   (f) true

   (g) true

   (h) true

   (i) true

   (j) true

10. (b) All red-haired people are not women.

    (c) A is an infinite set while B is a finite set.

    (d) 7 is an element of set A but not set B.

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13 + 0 = **13**

8 − **0** = 8

0 + 5 = 5

2 − 0 = 2

12 − 0 = **12**

0 + **3** = 3

Page 29

12 × 0 = **0**

0 × 3 = 0

0 × **8** = 0
↑
or any other number

6 × **0** × 4 × 9 = 0

3 × 8 × 0 = **0**

3 × 4 × 6 × **0** = 0

$\frac{0}{8} = 0$

$\frac{0}{2} = 0$

$\frac{0}{5} = 0$

or any other number except zero

$0 \div 9 = 0$

$0 \div 7 = 0$

$0 \div 11 = 0$

Page 30

$\frac{3}{0} =$ **undefined**

or any other number

$\frac{9}{0} =$ undefined

$\frac{6}{0} =$ undefined

$8 \div 0 =$ **undefined**

$17 \div$ undefined

$4 \div 0 =$ undefined

Page 32

1. $12 < 13$
2. $1 > 0$
3. $110 < 120$
4. $20 < 200$
5. $1001 > 1000$
6. $569 > 568$

Page 33

(i) $7 + 3 = 10$

(ii) $15 + 8 = 23$

(iii) $4 + 7 = 11$

(iv) $3 + 0 = 3$

(i) $7 \times 3 = 21$

(ii) $15 + 8 = 120$

(iii) $1 \times 9 = 9$

(iv) $0 \times 5 = 0$

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$6 + 9 = 9 + 6$

$2 + 12 = 12 + 2$

$r + s = s + r$

$2 + (6 + 3) = 2 + (3 + 6)$

$(a + b)(c + d) = (b + a)(d + c)$

$3x + 2y = 2y + 3x$

$3 \times 4 = 4 \times 3$

$4 \times m = m \times 4$

$3 + (2 \times 7) = 3 + (7 \times 2)$

$r(s + t) = (s + t)r$

$ab + cd = ba + dc$

$3 \times (2 + 6) = 3 \times (6 \times 2)$

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$4 + (7 + 9) = (4 + 7) + 9$

$(x + y) + 3z = x + (y + 3z)$

$(2a + 3b) + 5c = 2a + (3b + 5c)$

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$(8 \times 3) \times 4 = 8 \times (3 \times 4)$

$4 \times (x + y) = (4 \times x) \times y$

$(2 \times 3) \times 5 = 2 \times (3 \times 5)$

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1. $$\begin{aligned} 8 \times 65 &= 8(60 + 5) \\ &= (8 \times 60) + (8 \times 5) \\ &= 480 + 40 \\ &= 520 \end{aligned}$$

2. $$\begin{aligned} 7 \times 135 &= 7(100 + 30 + 5) \\ &= (7 \times 100) + (7 + 30) + (7 \times 5) \\ &= 700 + 210 + 35 \\ &= 945 \end{aligned}$$

Page 38

1. $6x + 5x = (6 + 5)x = 11x$

2. $10ab + 4ab = (10 + 4)ab = 14ab$

3. $9m + m = (9 + 1)m = 10m$

4. $a^2 + 2a^2 = (1 + 2)a^2 = 3a^2$

$9 + 0 = 9$ $3 \times 1 = 3$

$0 + 6 = 6$ $1 \times 8 = 8$

$2 + 0 = 2$ $1 \times 9 = 9$

Pages 40-43

1. (a) zero (e) closure, addition

(b) commutative (f) additive

(c) commutative and associative (g) distributive

(d) associative, multiplication

2. (a) true (f) true

(b) true (g) true

(c) false (h) false

(d) false (i) true

(e) true

3. Yes. Every natural number is also a whole number.

4. (a) $1 \times 3 = 3$ $3 \times 7 = 21$

$3 \times 5 = 15$ $5 \times 11 = 55$ Yes

Is set O closed under multiplication? Yes

(b) $5 + 7 = 12$ $7 + 9 = 16$ Any other sum.

$9 + 15 = 24$ $15 + 21 = 36$ No

Is set O closed under addition? No

(c) $6 \times 8 = 48$ $4 \times 6 = 24$ Any other product.

$12 + 10 = 120$ $10 \times 8 = 80$ Yes

Is set E closed under multiplication? Yes

5. (b) associative property of multiplication
   (c) commutative property of multiplication
   (d) additive identity property
   (e) commutative property of addition
   (f) multiplicative identity property
   (g) closure property of addition
   (h) associative property of addition
   (i) commutative property of addition
   (j) distributive property
   (k) commutative property of multiplication
   (l) additive identity property
   (m) commutative property of multiplication
   (n) associative property of addition

6. (a) 83
   (b) 0
   (c) 7
   (d) $11x$
   (e) 0
   (f) undefined
   (g) 14
   (h) 0
   (i) $3a + 3b$
   (j) $6y$
   (k) $\frac{0}{7} = 0$
   (l) $\frac{6}{3} = 2$
   (m) $0 \times 0 = 0$
   (n) undefined
   (o) $3 + 0 = 3$
   (p) $6 - 0 = 6$
   (q) $\frac{0}{9} = 0$
   (r) $(4 \times 8)m = 12m$

END OF LESSON A

LESSON B

Page 3

$|6| = \mathbf{6}$, $|5| = \mathbf{5}$, $|0| = \mathbf{0}$, $|-12| = \mathbf{12}$

What is the additive inverse of -3? $\mathbf{3}$, of 6? $\mathbf{-6}$, of 1? $\mathbf{-1}$

Page 5

$7 \times 2 = \mathbf{14}$

$-9 \times -9 = \mathbf{81}$

$-10 \times -1 = \mathbf{10}$

$4 \times 2 = \mathbf{8}$

$-4 \times -2 = \mathbf{8}$

$3 \times 4 = \mathbf{12}$

$-5 \times 8 = \mathbf{-40}$

$4 \times -2 = \mathbf{-8}$

$-3 \times 1 = \mathbf{-3}$

$7 \times -3 = \mathbf{-21}$

$-2 \times 2 = \mathbf{-4}$

$5 \times -1 = \mathbf{-5}$

Page 6

	Are the signs of the factors alike or different?	**Is the product positive or or negative?**	**Product**
-3×7	different	negative	−21
-2×-8	**alike**	**positive**	**16**
2×9	**alike**	**positive**	**18**
6×-8	**different**	**negative**	**−48**
-3×-3	**alike**	**positive**	**9**
-3×11	**different**	**negative**	**−33**
-1×6	**different**	**negative**	**−6**
-1×-1	**alike**	**positive**	**1**

Page 7

	Even or odd number of negative factors?	Is the product positive or negative?	Product
$2 \times -3 \times 4 \times -2$	even	positive	48
$-1 \times -1 \times -1 \times -1 \times -1 \times -1$	even	positive	1
$-1 \times -2 \times -3 \times -4 \times -5$	odd	negative	−120
$6 \times 3 \times -2 \times 1$	odd	negative	−36
−1 used as a factor 7 times	odd	negative	−1
−1 used as a factor 10 times	even	positive	1
$-3 \times -3 \times -3 \times -3$	even	positive	81
$-5 \times -2 \times 3 \times 2$	even	positive	60

$-24 \div -6 = 4$

$9 \div 1 = 9$

$\frac{-35}{-7} = 5$

$\frac{-10}{-2} = 5$

$22 \div 2 = 11$

$-8 \div -8 = 1$

$\frac{28}{7} = 4$

$\frac{24}{8} = 3$

Page 8

$14 \div -2 = -7$

$-2 \div 2 = -1$

$\frac{-9}{3} = -3$

$\frac{21}{-3} = -7$

$-36 \div 9 = -4$

$-15 \div 3 = -5$

$\frac{4}{-1} = -4$

$\frac{8}{-8} = -1$

1. $-32 \div -4 = \mathbf{8}$
2. $81 \div 9 = \mathbf{9}$
3. $-7 \div 1 = \mathbf{-7}$
4. $\frac{16}{-4} = \mathbf{-4}$
5. $\frac{-42}{-6} = \mathbf{7}$
6. $\frac{-10}{2} = \mathbf{-5}$
7. $\frac{3}{-1} = \mathbf{-3}$
8. $63 \div -7 = \mathbf{-9}$
9. $\frac{-1}{-1} = \mathbf{1}$

$5 + 7 = \mathbf{12}$

$4 + 3 = \mathbf{7}$

$13 + 2 = \mathbf{15}$

$1 + 2 = \mathbf{3}$

$9 + 1 = \mathbf{10}$

$8 + 8 = \mathbf{16}$

Page 9

$(-12) + (-2) = \mathbf{-14}$

$(-4) + (-4) = \mathbf{-8}$

$(-6) + (-1) = \mathbf{-7}$

$(-15) + (-6) = \mathbf{-21}$

$(-3) + (-1) = \mathbf{-4}$

$(-7) + (-6) = \mathbf{-13}$

1. $6 + (-5) = \mathbf{1}$
2. $-8 + 8 = \mathbf{0}$
3. $-9 + 5 = \mathbf{-4}$
4. $-3 + 10 = \mathbf{7}$
5. $2 + (-4) = \mathbf{-2}$
6. $5 + (-5) = \mathbf{0}$

Page 10

	Are the signs alike or different?	Must you add or subtract the absolute values?	Is the result positive or negative?	Result
$(-4) + (-5)$	alike	add	negative	-9
$5 + (-7)$	different	subtract	negative	-2
$-9 + 11$	different	subtract	positive	2
$-1 + 1$	different	subtract	neither	0
$2 + 3$	alike	add	positive	5
$(-7) + (-2)$	alike	add	negative	-9
$2 + (-2)$	different	subtract	neither	0
$-7 + 6$	different	subtract	negative	-1
$8 + (-5)$	different	subtract	positive	3
$(-1) + (-4)$	alike	add	negative	-5

Page 11

3. $12 - 20 = 12 + (-20)$

4. $1 - (-4) = 1 + 4$

5. $-4 - 5 = -4 + (-5)$

6. $2 - (-1) = 2 + 1$

7. $2 - 9 = 2 + (-9)$

8. $-3 - 8 = (-3) + (-8)$

Subtraction question	Corresponding addition question	Are the signs alike or different?	Sum
12 − (-7)	12 + 7	alike	19
−5 − 9	−5 + (−9)	alike	−14
−2 − (−6)	−2 + 6	different	4
−7 − 8	−7 + (−8)	alike	−15
−5 − (−1)	−5 + 1	different	−4
2 − 10	2 + (−10)	different	−8
−3 − 6	−3 + (−6)	alike	−9
8 − 9	8 + (−9)	different	−1
4 − (−3)	4 + 3	alike	7

Page 14

For any integers a, b, c:	Addition	Multiplication
Closure Properties	(a + b) is an integer.	(a × b) is an integer.
Commutative Properties	a + b = b + a	a × b = b × a
Associative Properties	a + (b + c) = (a + b) + c	a × (b × c) = (a × b) × c
Distributive Property	a(b + c) = ab + ac	
Identity Elements	Additive identity is 0. a + 0 = a	Multiplicative identity is 1. a × 1 = a
Inverse Element	Additive inverse of a is −a. a + (−a) = 0	The inverse element does not exist for integers.

Pages 15-16

1. $6 - 5 = 1$ $-4 - 5 = -9$

$7 - 9 = -2$ $9 - 9 = 0$

$3 - (-1) = 4$ $-2 - (-1) = -1$

Yes

2. $8 - 5 = 3$ and $5 - 8 = -3$

No

3. $= 6 - 5$ $= -2 - 3$

$= 1$ $= -5$

No

4. $3(5 - 8)$ $= (3 \times 5) - (3 \times 8)$

$= 3(-3)$ $= 15 - 24$

$= -9$ $= -9$

Yes

Pages 16-19 **Exercise – The Set of Integers**

1. (a) negative, left (b) 0

(c) greater , right (d) absolute value , 7

(e) additive inverse (f) positive

(g) negative (h) -3 and 3

(i) negative (j) negative

(k) negative (l) subtraction

(m) additive , multiplicative (n) subtraction, , division

(o) addition , inverse (p) distributes

2. (a) -8 (b) 3 (c) 7 (d) -7
(e) 40 (f) -9 (g) -15 (h) -5
(i) -7 (j) 9 (k) -39 (l) -5
(m) -6 (n) 16 (o) 8 (p) 0
(q) -8 (r) -14 (s) 100 (t) -13
(u) -5 (v) 4 (w) -5 (x) -1
(y) -60 (z) 27

3. (a) $-5 + -7 = -12$
(b) $-5(-4) = 20$
(c) $\frac{-9 + 5}{2} = \frac{-4}{2} = -2$
(d) $\frac{(5 - 9)}{-2} = \frac{-4}{-2} = 2$
(e) $\frac{(-1)(-9)}{3} = \frac{9}{3} = 3$
(f) $5 - (-6) - 15 = -4$
(g) $\frac{-18}{9} + 2 = -2 + 2 = 0$
(h) $-12 + 1 = -11$
(i) $\frac{8}{-4} = -2$
(j) 11

4.

(a)	$5 + -5 = 0$	additive inverse property
(b)	$(-8) + 7 = 7 + -8$	commutative property of addition
(c)	$4 \times (-6 \times 5) = (4 \times -6) \times 5$	associative property of multiplication
(d)	$(4 \times -3) \in \mathbf{I}$	closure property of multiplication
(e)	$1 \times -6 = -6$	multiplicative identity property
(f)	$-8 + 0 = -8$	additive identity property
(g)	$-2(-3 + 5) = (-2 \times -3) + (-2 \times 5)$	distributive property
(h)	$-7 + 7 = 0$	additive inverse property
(i)	$-3 \times -9 = -9 \times -3$	commutative property of multiplication
(j)	$(-5 + 2) \in \mathbf{I}$	closure property of addition
(k)	$-7 + 9 = 9 + (-7)$	commutative property of addition
(l)	$0 + (-3) = -3$	additive identity property
(m)	$1 \times 4 = 4$	multiplicative identity property
(n)	$3(4 + 7) = (3 \times 4) + (3 \times 7)$	distributive property
(o)	$(-8 + 4) + 3 = -8 + (4 + 3)$	associative property of addition

Page 21

1. $\frac{12}{-2} = -6$
2. $\frac{35}{7} = 5$
3. $\frac{-9}{1} = -9$
4. $\frac{-100}{-10} = 10$
5. $\frac{0}{7} = 0$
6. $\frac{-42}{-6} = 7$
7. $\frac{-8}{-8} = 1$
8. $\frac{625}{-25} = -25$
9. $\frac{11}{-1} = -11$

Page 22

1. $\frac{-7}{-8} = \frac{7}{8}$
2. $\frac{-6}{5} = \frac{-6}{5}$
3. $\frac{5}{-9} = \frac{-5}{9}$
4. $\frac{-9}{-11} = \frac{9}{11}$
5. $\frac{3}{-7} = \frac{-3}{7}$
6. $\frac{-3}{4} = \frac{-3}{4}$

Page 23

Fraction Pair	Cross Products	Equal or Not Equal?
$\frac{-15}{20}, \frac{-3}{4}$	$-15 \times 4 = -60$ $20 \times -3 = -60$	equal
$\frac{3}{8}, \frac{12}{33}$	$3 \times 33 = 99$ $8 \times 12 = 96$	not equal
$\frac{8}{36}, \frac{2}{9}$	$8 \times 9 = 72$ $36 \times 2 = 72$	equal
$\frac{2}{3}, \frac{-4}{6}$	$2 \times 6 = 12$ $3 \times -4 = -12$	not equal
$\frac{-4}{10}, \frac{-14}{35}$	$-4 \times 35 = -140$ $10 \times -14 = -140$	equal

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1. $\frac{-3}{4} = \frac{-24}{32}$
2. $\frac{5}{4} = \frac{10}{8}$
3. $\frac{4}{5} = \frac{16}{20}$
4. $\frac{-2}{3} = \frac{-10}{15}$
5. $\frac{1}{4} = \frac{16}{64}$
6. $5 = \frac{15}{3}$

1. $\frac{30}{66} = \frac{5}{11}$
2. $\frac{-40}{56} = \frac{-5}{7}$
3. $\frac{10}{100} = \frac{1}{10}$
4. $\frac{14}{21} = \frac{2}{3}$
5. $\frac{64}{48} = \frac{4}{3}$
6. $\frac{-36}{60} = \frac{-3}{5}$

Page 27

2. $\frac{\overset{3}{\cancel{6}}}{\underset{5}{\cancel{45}}} \times \frac{\overset{1}{\cancel{9}}}{\underset{11}{\cancel{22}}} = \frac{3 \times 1}{5 \times 11} = \frac{3}{55}$

3. $\frac{\overset{11}{\cancel{33}}}{\underset{2}{\cancel{8}}} \times \frac{\overset{1}{\cancel{4}}}{\underset{5}{\cancel{15}}} = \frac{11 \times 1}{2 \times 5} = \frac{11}{10}$

4. $\frac{\overset{-1}{\cancel{-14}}}{\underset{5}{\cancel{15}}} \times \frac{\overset{-3}{\cancel{-9}}}{\underset{4}{\cancel{56}}} = \frac{-1 \times -3}{5 \times 4} = \frac{3}{20}$

Page 28

1. $\frac{5}{8} \times \frac{1}{5} = \frac{1 \times 1}{8 \times 1} = \frac{1}{8}$

2. $\frac{\overset{-2}{\cancel{-14}}}{9} \times \frac{5}{\underset{1}{\cancel{7}}} = \frac{-2 \times 5}{9 \times 1} = \frac{-10}{9}$

3. $\frac{26}{9} \times \frac{27}{-2} = \frac{13 \times 3}{1 \times -1} = -39$

4. $\frac{11}{\underset{9}{\cancel{27}}} \times \frac{\overset{1}{\cancel{3}}}{1} = \frac{11 \times 1}{9 \times 1} = \frac{11}{9}$

Page 30

$$= 1 \div \frac{-15}{2} = 1 \times \frac{-2}{15} = \frac{-2}{15}$$

Page 32

1. $<$, left
2. $>$, right
3. $<$, left

Page 35

Closure Properties	$\left(\frac{a}{b}+\frac{c}{d}\right)$ is a rational number.	$\left(\frac{a}{b}\times\frac{c}{d}\right)$ is a **rational** number.
Commutative Properties	$\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$	$\frac{a}{b}\times\frac{c}{d}=\frac{c}{d}\times\frac{a}{b}$
Associative Properties	$\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)=\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}$	$\frac{a}{b}\left(\frac{c}{d}\times\frac{e}{f}\right)=\left(\frac{a}{b}\times\frac{c}{d}\right)\frac{e}{f}$
Distributive Property	$\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right)=\left(\frac{a}{b}\times\frac{c}{d}\right)+\left(\frac{a}{b}\times\frac{e}{f}\right)$	
Identity Elements	Additive identity is 0. $\frac{a}{b}+0=\frac{a}{b}$	Multiplicative identity is 1. $\frac{a}{b}\times 1=\frac{a}{b}$
Inverse Elements	Additive inverse of $\frac{a}{b}$ is $\frac{-a}{b}$. $\frac{a}{b}+\frac{-a}{b}=0$	Multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$. $\frac{a}{b}\times\frac{b}{a}=1$

Pages 36-40

1. (a) multiplied, divided
   (b) subset
   (c) right
   (d) dense
   (e) complete
   (f) least common
   (g) multiplicative, product
   (h) multiplication, reciprocal or multiplicative inverse, $\frac{3}{5}$
   (i) positive, negative, additive
   (j) zero

2. (a) $\frac{-3}{28}+\frac{17}{28}=\frac{-3+17}{28}=\frac{14}{28}=\frac{1}{2}$

   (b) $\frac{7}{8}-\frac{5}{8}=\frac{7-5}{8}=\frac{2}{8}=\frac{1}{4}$

   (c) $\frac{-1}{4}+\frac{-3}{4}=\frac{-1-3}{4}=\frac{-4}{4}=-1$

(d) $\frac{3}{25}+\frac{9}{75}=\frac{3\times 3}{25\times 3}+\frac{9}{75}=\frac{9+9}{75}=\frac{18}{75}=\frac{6}{25}$

(e) $\frac{7}{24}-\frac{5}{18}=\frac{7\times 3}{24\times 3}-\frac{5\times 4}{18\times 4}=\frac{21-20}{72}=\frac{1}{72}$

(f) $\frac{5}{8}-\frac{11}{12}=\frac{5\times 3}{8\times 3}-\frac{11\times 2}{12\times 2}=\frac{15-22}{24}=\frac{-7}{24}$

(g) $\frac{-27}{9}\times\frac{14}{35}=\frac{\overset{-3}{\cancel{-27}}}{\underset{1}{\cancel{9}}}\times\frac{\overset{2}{\cancel{14}}}{\underset{5}{\cancel{35}}}=\frac{-3\times 2}{1\times 5}=\frac{-6}{5}$

(h) $\frac{25}{8}\times\frac{24}{5}=\frac{\overset{5}{\cancel{25}}}{\underset{1}{\cancel{8}}}\times\frac{\overset{3}{\cancel{24}}}{\underset{1}{\cancel{5}}}=5\times 3=15$

(i) $-12\times\frac{-3}{8}=-\overset{-3}{\cancel{12}}\times\frac{-3}{\underset{2}{\cancel{8}}}=\frac{-3\times -3}{2}=\frac{+9}{2}$

(j) $\frac{3}{8}\div 4=\frac{3}{8}\times\frac{1}{4}=\frac{3}{32}$

(k) $\frac{-2}{3}\div\frac{4}{9}=\frac{\overset{-1}{\cancel{-2}}}{\underset{1}{\cancel{3}}}\times\frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{4}}}=\frac{-1\times 3}{1\times 2}=\frac{-3}{2}$

(l) $\frac{4}{15}\div\frac{14}{45}=\frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{15}}}\times\frac{\overset{3}{\cancel{45}}}{\underset{7}{\cancel{14}}}=\frac{2\times 3}{1\times 7}=\frac{6}{7}$

(m) $\overset{1}{\cancel{3}}\times\frac{\overset{1}{\cancel{7}}}{\underset{\underset{1}{\cancel{3}}}{\cancel{6}}}\times\frac{\overset{2}{\cancel{4}}}{\underset{3}{\cancel{21}}}=\frac{1\times 1\times 2}{1\times 3}=\frac{2}{3}$

(n) $\frac{2}{3}+\frac{5}{6}+\frac{7}{12}=\frac{2\times 4}{3\times 4}+\frac{5\times 2}{6\times 2}+\frac{7}{12}=\frac{8+10+7}{12}=\frac{25}{12}$

(o) $5\div\left(\frac{1}{2}+\frac{1}{3}\right)=5\div\left(\frac{3}{6}+\frac{2}{6}\right)=5\div\frac{5}{6}=\overset{1}{\cancel{5}}\times\frac{6}{\underset{1}{\cancel{5}}}=6$

(p) $=\frac{3}{4}\div\frac{1}{6}=\frac{3}{\underset{2}{\cancel{4}}}\times\frac{\overset{3}{\cancel{6}}}{1}=\frac{3\times 3}{2}=\frac{9}{2}$

(q) $=\frac{3}{4}\div\left(\frac{1}{2}-\frac{5}{2}\right)=\frac{3}{4}\div\frac{-4}{2}=\frac{3}{\underset{2}{\cancel{4}}}\times\frac{\overset{1}{\cancel{2}}}{-4}=\frac{-3}{8}$

(r) $-2\div\frac{1}{5}=-2\times\frac{5}{1}=-10$

3. (a) 6 (b) -2 (c) -16

(d) 63 (e) 12 (f) 3

4. (b) $-1, -\frac{1}{4}, 0, \frac{1}{4}$ (c) $-1, -\frac{1}{4}, 0, \frac{1}{4}, 1$ (d) ϕ

(e) $-\frac{1}{4}, 0, \frac{1}{4}$ (f) $\frac{1}{4}$

5.
$$\frac{2}{8} - \frac{1}{8} = \frac{1}{8} \quad \Bigg| \quad \frac{5}{8} - \frac{2}{8} = \frac{3}{8}$$

No

6.
$$\frac{3}{4} - \frac{5}{8} = \frac{6-5}{8} = \frac{1}{8} \quad \Bigg| \quad \frac{8}{5} - \frac{3}{4} = \frac{5-6}{8} = -\frac{1}{8}$$

No

7.
$$\left(\frac{2}{3} \times \frac{5}{3}\right) \div \frac{5}{8} = \frac{\overset{2}{\cancel{10}}}{9} \times \frac{8}{\underset{1}{\cancel{5}}} = \frac{16}{9}$$

$$\frac{2}{3} \div \left(\frac{3}{5} \div \frac{5}{8}\right) = \frac{2}{3} \div \left(\frac{3}{5} \times \frac{8}{5}\right) = \frac{2}{3} \div \frac{24}{25} = \frac{\overset{1}{\cancel{2}}}{3} \times \frac{25}{\underset{12}{\cancel{24}}} = \frac{25}{36}$$

No

8.
$$\frac{5}{8} \times \frac{16}{25} = \frac{2}{5} \quad \Bigg| \quad \frac{25}{16} \div \frac{5}{8} = \frac{\overset{5}{\cancel{25}}}{\underset{2}{\cancel{16}}} \times \frac{\overset{1}{\cancel{8}}}{\underset{1}{\cancel{5}}} = \frac{5}{2}$$

No

9.

		Additive Inverse	Multiplicative Inverse
(b)	$\frac{-1}{2}$	$\frac{1}{2}$	-2
(c)	3	-3	$\frac{1}{3}$
(d)	$\frac{8}{5}$	$\frac{-8}{5}$	$\frac{5}{8}$
(e)	0	0	none

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1. addition
2. multiplication
4. Multiplication is commutative.
5. Addition is associative.
7. Multiplication distributes over addition.
8. Additive identity is 0.
9. Multiplicative identity is 1.
10. Additive inverse of "a" is −a.
11. Multiplicative inverse of "a" is $\frac{1}{a}$.

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1. Yes
2. No
3. Yes
4. No
5. Yes
6. Yes

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Exercise – The Set of Real Numbers

1. (a) complete
   (b) disjoint, real
   (c) commutative
   (d) multiplicative identity
   (e) real, closed
   (f) rational, real
   (g) irrational
   (h) terminating, non-terminating repeating
   (i) additive
   (j) rational, irrational

2. (a) $1.\overline{6}$
   (b) 0.006
   (c) $2.1\overline{3}$
   (d) 1.625

3. (a) $\frac{35}{100} = \frac{7}{20}$
   (b) $\frac{8}{10000} = \frac{1}{1250}$

4. (a) Let $x = .\overline{24} = .2424\ldots$

$$\begin{aligned} \text{Then } 100x &= 24.2424\ldots \\ x &= .2424\ldots \\ \hline 99x &= 24 \\ x &= \frac{24}{99} = \frac{8}{33} \end{aligned}$$

   (b) Let $x = 4.\overline{5} = 4.555\ldots$

$$\begin{aligned} \text{Then } 10x &= 45.555\ldots \\ x &= 4.555\ldots \\ \hline 9x &= 41 \\ x &= \frac{41}{9} \end{aligned}$$

5. (a) $= 5 + 4.242 = 9.242$ (b) $= \frac{8 - 6.245}{3} = \frac{1.755}{3} = 0.585$

(c) $= \frac{3}{2}(6.856) = 10.284$ (d) $= \frac{2.236 + 3.606}{2} = \frac{5.842}{2} = 2.921$

6.

		N	W	I	Q	R
(a)	-5			✓	✓	✓
(b)	2	✓	✓	✓	✓	✓
(c)	$\frac{-5}{9}$				✓	✓
(d)	$0.3654\ldots$					✓
(e)	$\sqrt{16}$	✓	✓	✓	✓	✓
(f)	$\sqrt{2}$					✓
(g)	π					✓
(h)	-10.1				✓	✓
(i)	0		✓	✓	✓	✓
(j)	$0.\overline{15}$				✓	✓
(k)	$\sqrt{-5}$					
(l)	$\frac{2}{3}$				✓	✓

7. (a) true (b) true (c) false (d) false

(e) true (f) true (g) true (h) false

(i) true (j) false (k) true (l) false

(m) true (n) true (o) false (p) false

(q) true

END OF LESSON B

LESSON C

Page 2

Similarly $(-2)^6 = 64$, while $-2^6 = -64$

Similarly $\left(\frac{4}{5}\right)^3 = \frac{64}{125}$, while $\frac{4^3}{5} = \frac{64}{5}$

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Thus $(-14)^0 = 1$, $2^0 = 1$, $\left(\frac{-2}{3}\right)^0 = 1$ and $100^0 = 1$

Page 10

1. $5^{-3} = \frac{1}{5^3}$

2. $(-2)^{-9} = \frac{1}{(-2)^9}$

3. $\frac{1}{8^{-6}} = \frac{1}{\frac{1}{8^6}} = 1 \div \frac{1}{8^6} = 1 \times \frac{8^6}{1} = 8^6$

4. $\frac{1}{4 \times 3^{-2}} = \frac{1}{4 \times \frac{1}{3^2}} = 1 \div \frac{4}{3^2} = 1 \times \frac{3^2}{4} = \frac{3^2}{4}$

Pages 13-15

Exercise – Powers

1. (a) base, factor (b) positive integral, base, positive

(c) negative, fractional (d) negative, odd

(e) common, base, exponents (f) zero

(g) reciprocal or multiplicative inverse (h) subtract

2. (a) 2^{-6} (b) 2^2 (c) 2^5

(d) 2^{-1} (e) 2^0 (f) 2^{-3}

3. (a) 5^2 (b) 5^{-1} (c) 5^{-3}

(d) 5^4 (e) 5^{-2} (f) 5^0

4. (a) $\frac{1}{5^8}$ (b) $\frac{6^7}{4^6}$ (c) 3^9

(d) 7×4^3 (e) $\frac{12^8}{10^7}$ (f) $\frac{1}{6^8} + \frac{1}{4^2}$

5. (a) -64 (b) $-\frac{1}{32}$ (c) 0.0004

(d) 1 (e) $\frac{1}{64}$ (f) $\frac{1}{81}$

(g) 27 (h) $\frac{3}{2}$ (i) 216

(j) $1 \times 64 = 64$ (k) $5^{10-9-1} = 5^{10-10} = 5^0 = 1$

(l) $-1 \times 4 = -4$ (m) $4^{0-(-2)} = 4^{0+2} = 4^2 = 16$

(n) $\frac{4}{3} - \frac{4}{9} = \frac{12-4}{9} = \frac{8}{9}$ (o) $-27 - 4 = -31$

(p) $-2(9 - 25) = -2(-16) = +32$

6. (a) $\frac{1}{4^9}$ (b) $6^{-5-11} = 6^{-16}$

(c) $(\sqrt{5})^{7-6} = \sqrt{5}$ (d) $(-8)^{14-4} = (-8)^{10}$

(e) $4^{5-9} = 4^{-4}$ (f) $3^{3-(-7)} = 3^{3+7} = 3^{10}$

(g) $5^{3+5-2} = 5^6$ (h) $2^{0+5-12} = 2^{-7}$

(i) $(-3)^{6+8-(-4)-7} = (-3)^{11}$ (j) $3^{1-4-6} = 3^{-9}$

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Exercise – Roots of Real Numbers

1. (a) positive, absolute, sign (b) principal, negative

(c) perfect square (d) radicands, square root

(e) indentical (f) negative

(g) mixed radical

Pages 26-27

1. (h) rationalize, $\sqrt{5}$

2. (a) 100×3 (b) 16×10 (c) 4×11

(d) 64×2 (e) 49×5 (f) 121×3

3. (a) $5\sqrt{3}$ (b) $\sqrt{81 \times 2} = 9\sqrt{2}$ (c) $\sqrt{100 \times 10} = 10\sqrt{10}$

(d) $\frac{\sqrt{1}}{\sqrt{8}} = \frac{1 \quad \times\sqrt{2}}{2\sqrt{2} \quad \times\sqrt{2}} = \frac{\sqrt{2}}{4}$ (e) $\sqrt{\frac{3}{2}} = \frac{\sqrt{3} \quad \times\sqrt{3}}{\sqrt{2} \quad \times\sqrt{3}} = \frac{\sqrt{6}}{2}$

(f) $\frac{6}{5\sqrt{12}} = \frac{6 \quad \times\sqrt{3}}{5 \times 2\sqrt{3} \quad \times\sqrt{3}} = \frac{6\sqrt{3}}{10 \times 3} = \frac{\sqrt{3}}{5}$

(g) $\frac{-2\sqrt{4} \quad \times\sqrt{5}}{\sqrt{5} \quad \times\sqrt{5}} = \frac{-2(2) \times \sqrt{5}}{5} = \frac{-4\sqrt{5}}{5}$

(h) $-3\sqrt{16 \times 5} = -3 \times 4\sqrt{5} = -12\sqrt{5}$

4. (a) -5 (b) $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$ (c) 6

(d) $4 \times 3 = 12$

(e) $4 \times -3\sqrt{3 \times 15} = -12\sqrt{45} = -12 \times 3\sqrt{5} = -36\sqrt{5}$

(f) $2 \times 5 \times \sqrt{\frac{3}{5} \times \frac{5}{3}} = 10\sqrt{1} = 10 \times 1 = 10$

(g) $-\sqrt{3 \times 2 \times 6} = -\sqrt{36} = -6$

(h) $-3 \times 4 \times 1\sqrt{2 \times 2 \times 7} = -12 \times 2\sqrt{7} = -24\sqrt{7}$

(i) $\frac{-1 \quad \times\sqrt{5}}{3\sqrt{5} \quad \times\sqrt{5}} = \frac{-\sqrt{5}}{3 \times 5} = \frac{-\sqrt{5}}{15}$

(j) $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$

(k) $\frac{15\sqrt{10}}{5\sqrt{2}} = \frac{15}{5}\sqrt{\frac{10}{2}} = 3\sqrt{5}$

(l) $2\sqrt{\frac{6}{10}} = 2\sqrt{\frac{3}{5}} = \frac{2\sqrt{3} \quad \times\sqrt{5}}{\sqrt{5} \quad \times\sqrt{5}} = \frac{2\sqrt{15}}{5}$

(m) $\sqrt{\frac{2}{3}} + 5\sqrt{\frac{2}{3}} = (1 + 5)\sqrt{\frac{2}{3}} = \frac{6\sqrt{2}}{\sqrt{3}}\,\frac{\times \sqrt{3}}{\times \sqrt{3}} = \frac{6\sqrt{6}}{3} = 2\sqrt{6}$

(n) $(3 + 4 - 9)\sqrt{13} = -2\sqrt{13}$

(o) $\sqrt{3} - 16\sqrt{3} + 15\sqrt{3} = (1 - 16 + 15)\sqrt{3} = 0\sqrt{3} = 0$

(p) $-5\sqrt{4 \times 11} - \sqrt{9 \times 11} + 6\sqrt{11} = -5 \times 2\sqrt{11} - 3\sqrt{11} + 6\sqrt{11}$
$= (-10 - 3 + 6)\sqrt{11} = -7\sqrt{11}$

(q) $10\sqrt{4 \times 7} + 4\sqrt{9 \times 7} - 5\sqrt{16 \times 7}$
$= 10 \times 2\sqrt{7} + 4 \times 3\sqrt{7} = 5 \times 4\sqrt{7}$
$= 20\sqrt{7} + 12\sqrt{7} - 20\sqrt{7}$
$= (20 + 12 - 20)\sqrt{7}$
$= 12\sqrt{7}$

Pages 36-39 **Exercise – Operating with Variable Terms**

1. (a) variable
(b) square, increase, 1
(c) decrease, 5, multiply, -2
(d) domain
(e) factors, -5, ab
(f) product, powers
(g) quotient, powers
(h) like, literal
(i) numerical, literal

2. (a) $8m$
(b) $-2y$
(c) $4a^2d$
(d) 0
(e) $2y^2$
(f) $-18w$
(g) $-5\,mn$
(h) 0
(i) $-2b + 4a$
(j) $5p^2 + 5pq - 3q^2$
(k) $-x - y$
(l) $3\sqrt{5}x$
(m) $-3a^2 - 11ab + 7b^2$
(n) $-4x + 2z$
(o) $m^2 - 5mn + 2n^2$
(p) $-4ab - 5ac + b^2 - 3c^2 + 10bc$

3. (a) y^6
(b) $-x^6$
(c) b^4
(d) $2x^3$
(e) $-12ab^3$
(f) $2x^2$
(g) $14x^3y^3$
(h) $-4ab^3c^5$
(i) $24r^3st^4$
(j) $10x^7y^5$

4. (a) y^6

(b) $\frac{1}{x^4}$

(c) $-mn$

(d) $\frac{1}{xy}$

(e) 4

(f) $\frac{-3y}{x}$

(g) $\frac{-2ac}{b}$

(h) $\frac{-2p}{q}$

(i) $3a$

(j) $\frac{3b^3c^3}{2}$

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1. $(9x)^2 = (9)^2\ x^2 = 81x^2$
2. $(-3abc)^3 = (-3)^3\ a^3\ b^3\ c^3 = -27a^3b^3c^3$
3. $(-5x)^4 = (-5)^4\ x^4 = 625x^4$
4. $(3m)^5 = 3^5\ m^5 = 243m^5$

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1. $\left(\frac{-4}{b}\right)^2 = \frac{(-4)^2}{b^2} = \frac{16}{b^2}$
2. $\left(\frac{m}{n}\right)^7 = \frac{m^7}{n^7}$
3. $\left(\frac{-y}{x}\right)^4 = \frac{y^4}{x^4}$
4. $\left(\frac{1}{a}\right)^3 = \frac{1}{a^3}$

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1. r^{15}
2. y^6
3. m^8
4. m^{12}

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Radical	Values of Variables	Substitution	Is Radical Now a Real Number?
$\sqrt{3ab}$	$a = 1, b = 5$	$\sqrt{3(1)(5)} = \sqrt{15}$	yes
$\sqrt{y}$	$y = -2$	$\sqrt{-2}$	no
$\sqrt{x + 1}$	$x = 0$	$\sqrt{0 + 1} = \sqrt{1} = 1$	yes
$\sqrt{x^2}$	$x = -2$	$\sqrt{(-2)^2} = \sqrt{4} = 2$	yes
$\sqrt{x^3}$	$x = -2$	$\sqrt{(-2)^3} = \sqrt{-8}$	no
$\sqrt{x^4}$	$x = -1$	$\sqrt{(-1)^4} = \sqrt{1} = 1$	yes
$\sqrt{a^2b}$	$a = -1, b = 2$	$\sqrt{(-1)^2(2)} = \sqrt{2}$	yes
$\sqrt{ab^2}$	$a = -1, b = 2$	$\sqrt{(-1)(2)^2} = \sqrt{-4}$	no
$\sqrt{xy}$	$x = -2, y = 3$	$\sqrt{-2(3)} = \sqrt{-6}$	no
$\sqrt{-2ab}$	$a = -3, b = 2$	$\sqrt{(-2)(-3)(2)} = \sqrt{12}$	yes

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1. $\sqrt{(ab)^2} = |ab|$
2. $\sqrt{(5y)^2} = |5y| = 5|y|$
3. $\sqrt{(4ab^2c^3)^2} = |4ab^2c^3| = 4b^2|ac^3|$

Pages 48-52

Exercise – Powers and Roots of Variable Terms

1. (a) negative, exponent, negative, reciprocal or multiplicative inverse
   (b) power, power
   (c) power, quotient, power, product
   (d) exponent, factor
   (e) positive
   (f) absolute

2. (a) $|x|$
   (b) $|mn|$
   (c) $\sqrt{(3y)^2} = |3y| = 3|y|$
   (d) $\sqrt{(10z)^2} = |10z| = 10|z|$
   (e) $\sqrt{5^6} = \sqrt{(5^3)^2} = 5^3$
   (f) $\sqrt{(y^2)^2} = y^2$
   (g) $\sqrt{(7x^5)^2} = |7x^5| = 7|x^5|$
   (h) $\sqrt{(2xy^3)^2} = |2xy^3| = 2|xy^3|$

3. (a) product of powers property
(b) power of a power property
(c) definition of zero exponent
(d) power of a product property
(e) power of a quotient property
(f) product of powers property
(g) definition of negative exponents
(h) power of a product property
(i) quotient of powers property
(j) definition of zero exponents
(k) definition of negative exponents
(l) power of a quotient property

4. (a) $\frac{x}{y^5}$
(b) $x^3 \times x^2 = x^5$
(c) $(m^3)^5 = m^{3\times 5} = m^{15}$
(d) $\frac{a^{-2}}{b^{-2}} = \frac{b^2}{a^2}$
(e) $(3b)^2 = 3^2b^2 = 9b^2$
(f) $x^{-1} + y^{-1} = \frac{1}{x} + \frac{1}{y} = \frac{y + x}{xy}$
(g) $\frac{x^4}{x^4} = x^{4-4} = x^0 = 1$
(h) $\frac{1}{5x^{-3}} = \frac{x^3}{5}$
(i) $(2x)^{-5} = \frac{1}{32x^5}$
(j) $(ab)^2 = a^2b^2$
(k) $(5y)^0 = 1$
(l) $\left(\frac{2x}{y}\right)^3 = \frac{8x^3}{y^3}$

5. (a) $32b^5$
(b) $-27y^3$
(c) $49m^2$
(d) $64a^6b^3$
(e) $\frac{y}{x}$
(f) $\frac{1}{(3x)^2} = \frac{1}{9x^2}$
(g) $\frac{3^4m^8}{n^4} = \frac{81m^8}{n^4}$
(h) $\frac{64x^6}{y^6}$
(i) $\frac{64a^6}{b^2c^2}$
(j) $\frac{b^2}{25a^2}$

6. (a) $12x^2$
(b) $-4y^{3+1} = -4y^4$
(c) $-3 \times 5y^{-3-1} = -15y^{-4}$
(d) $\frac{x^3y^3}{x^2 5} = x^{3-2}y^{3-1} = xy^2$

(e) $4a^2b^2 - 3a^2b^2 = (4 - 3)a^2b^2 = 1a^2b^2 = a^2b^2$

(f) $\frac{x^{15}}{x^{-4}} = x^{15-(-4)} = x^{15+4} = x^{19}$

(g) $\frac{8a^3}{b^3} \times \frac{a^2}{4b} = \frac{8}{4} \times \frac{a^{3+2}}{b^{3+1}} = \frac{2a^5}{b^4}$

(h) $\frac{-8x^6y^3}{x^{3-1}} = \frac{-8x^6y^3}{x^2} = -8x^{6-2}y^3 = -8x^4y^3$

7. (a) $\frac{5a}{b}$ (b) $\frac{2}{c^5d^2}$ (c) $\frac{x^5}{8}$

(d) $\frac{y^2}{3x}$ (e) $4x^3$

8. (a) (i) When $x = 5$, $\sqrt{x - 3} = \sqrt{5 - 3} = \sqrt{2}$ which is real.

(ii) When $x = 1$, $\sqrt{x - 3} = \sqrt{1 - 3} = \sqrt{-2}$ which is not real.

(b) (i) When $y = 1$, $\sqrt{y^3} = \sqrt{1^3} = \sqrt{1} = 1$ which is real.

(ii) When $y = -1$, $\sqrt{y^3} = \sqrt{(-1)^3} = \sqrt{-1}$ which is not real.

(c) (i) When $x = 2$, $\sqrt{2x + 1} = \sqrt{2(2) + 1} = \sqrt{5}$ which is real.

(ii) When $x = -2$, $\sqrt{2(-2) + 1} = \sqrt{-3}$ which is not real.

(d) (i) When $x = 3$, $\sqrt{3x^2} = \sqrt{3(3)^2} = \sqrt{27}$ which is real.

(ii) There is no real value of x which will make $\sqrt{3x^2}$ not real.

END OF LESSON C

LESSON D

Page 1

Example 1:

This polynomial consists of four terms. The first term, $\frac{3}{2}x$, is the product of the real number $\frac{3}{2}$ and the variable **x** raised to the first power. The second term, $5y^2$, is the product of the real number **5** and the variable **y** raised to the **second** power. The third term, $4xy^3$, is the product of the real number **4**, the variable x raised to the **first** power, and the variable y raised to the **third** power. The fourth term, 7 is a real number.

Example 2:

If this polynomial were written in the form $\sqrt{3}t^4 + (-7t^2)$, it could be regarded as a sum. This polynomial consists of **two** terms. The first term, $\sqrt{3}t^4$, is the product of the real number $\sqrt{3}$ and the variable t raised to the **fourth** power. The second term, $-7t^2$, is the product of the real number -7 and the variable **t** raised to the **second** power.

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1. binomial
2. monomial
3. trinomial
4. polynomial
5. binomial
6. polynomial
7. monomial
8. binomial
9. monomial
10. trinomial

Page 5

Which polynomial listed above is a trinomial? $\mathbf{2m^2 + m - 3}$

Which is a monomial? $\mathbf{x}$

Which is a binomial? $\mathbf{3t + 5}$

Give an example of a binomial in the one variable z. (Make up your own example.) $z^3 + 5$

How many different variables are involved in the third polynomial? **4 variables**

Which expression is a trinomial? $\mathbf{3tuv + w^2 + v^2}$

Give an example of a monomial in the three variables a, b, and c. $\mathbf{3a^3b^2c}$

There are other answers.

2. polynomial in one variable
3. polynomial in three variables
4. polynomial in two variables
5. polynomial in four variables
6. polynomial in three variables
7. polynomial in one variable

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1. degree 2
2. degree 6
3. degree 3
4. degree 1
5. degree 4
6. degree 2
7. degree 7
8. degree 2

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Could each of these constant polynomials be classified as a monomial? Yes

Which of these linear polynomials are monomials? $-3y$, x

Which is a trinomial? $\frac{2}{3}a - \frac{1}{2}b + c$

Which is a polynomial in two variables? $x - 5y$

Which is a linear binomial in one variable? $a + 3$

Which quadratic polynomials above are binomials? $7y^2 - 3y$, $2yz - y$, $2y^2 - 3$

Which is a quadratic trinomial in one variable? $12x^2 + 17x + 15$

Which is a quadratic trinomial in two variables? $a^2 + 3ab + b^2$

Which is a quadratic trinomial in three variables? $3a + 5bc + 6c$

Which is a quadratic monomial in two variables? $4xy$

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Which expressions above are cubic binomials in two variables? $4x^3 - 3xy^2$, $2a^2b + b$

Which expression above is a cubic trinomial in one variable? $2a^3 - 5a^2 + 3$

In the quartic polynomial $5a^2bc - 3a^2 + 2b^2c$ which term is of degree 4? $5a^2bc$

Which expression above is a quartic binomial in two variables? $2x^2y^2 + 8$

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3. No – The coefficient of y is $\frac{1}{2}$ which is not an integer.

Yes – All polynomials over the integers are also polynomials over the rationals because $I \subseteq Q$.

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$-6x + 6$, $a = -6$ and $b = 6$

$-y + 0$, $a = -1$ and $b = 0$

$-x^2 + 8x - 6$, $a = -1$, $b = 8$ and $c = -6$

$3y^2 - 3y$, $a = 3$, $b = -3$ and $c = 0$

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$y^3 + 6y^2 - 1$, $a = 1$, $b = 6$, $c = 0$ and $d = -1$

$-5m^4 - 4m^3 + 4m^2 + 6$, $a = -5$, $b = -4$, $c = 4$, $d = 0$ and $e = 6$

Pages 15-19

Exercise – Polynomials

1. (a) three, trinomial
   (b) variable
   (c) $-ax^3$, $3bx^2$, $-4x$, -1
   (d) 7
   (e) binomial
   (f) 6
   (g) linear
   (h) negative, real
   (i) highest
   (j) like, decrease
   (k) $ax^3 + bx^2 + cx + d$
   (l) quartic, four
   (m) -1, -8
   (n) quadratic, one
   (o) constant, zero

2.

	Polynomial	Number of Terms	Number of Variables	Degree
(a)	$x^3 - 3xy + 5y^2$	3 (trinomial)	2	3 (cubic)
(b)	$3x^2yz$	**1 (monomial)**	**3**	**4 (quartic)**
(c)	$5 + 2ab$	**2 (binomial)**	**2**	**2 (quadratic)**
(d)	-7	**1 (monomial)**	**none**	**0 (constant)**
(e)	$-2y + 3$	**2 (binomial)**	**1**	**1 (linear)**
(f)	$x^2 - 4xy + 2y^2$	**3 (trinomial)**	**2**	**2 (quadratic)**
(g)	$x^4 - x^2$	**2 (binomial)**	**1**	**4 (quartic)**
(h)	$-5a^3 + 2ab + 3b^2 - 7$	**4**	**2**	**3 (cubic)**
(i)	$5y^2 + 7$	**2 (binomial)**	**1**	**2 (quadratic)**
(j)	$x^4 - 5x^2 + 4$	**3 (trinomial)**	**1**	**4 (quartic)**
(k)	$n^2n^2 - 1$	**2 (binomial)**	**2**	**4 (quartic)**

3. $4x^{-2}$ – negative exponents

$\frac{1}{y} + 7$ – variable in the denominator

$5\sqrt{ab} + b - 7$ – square root of variable

$2xy + y + x^{\frac{1}{2}}$ – fractional exponents

$\frac{4x^3 + 2y^2}{z}$ – variable in the denominator

4. (a) $3a + b$
   (b) $a + b + c + d$
   (c) a^5
   (d) $x^3 + x + y$
   (e) 3
   (f) $3a + 1$
   (g) $3x^2 + x + 1$
   (h) a^4

These are examples, answers can vary.

5. (a) polynomial over the integers
   (b) polynomial over the rationals
   (c) polynomial over the reals
   (d) polynomial over the integers

6.

Polynomial in One Variable	Standard Form	Classification (Degree)	Numerical Coefficients				
			a	b	c	d	e
$x^2 - 3x^3 + 5$	$-3x^3 + x^2 + 5$	cubic $(ax^3 + bx^2 + cx + d)$	-3	1	0	5	–
$7 - 3x - 2x$	$-5x + 7$	linear $(ax + b)$	-5	7	–	–	–
$2y^3 - 3y^4 + y^2 - 1$	$-3y^4 + 2y^3 + y^2 - 1$	quartic $(ax^4 + bx^3 + cx^2 + dx + e)$	-3	2	1	0	-1
$8 - 3x^2 + 9$	$-3x^2 + 17$	quadratic $(ax^2 + bx + c)$	-3	0	17	–	–
$2x^2 - 5x^3 + 4$	$-5x^3 + 2x^2 + 4$	cubic $(ax^3 + bx^2 + cx + d)$	-5	2	0	4	–
$3 + 2y + 6y^2$	$6y^2 + 2y + 3$	quadratic $(ax^3 + bx^2 + cx + d)$	6	2	3	–	–
$8y^3 + 3y - 5 + 2y^2$	$8y^3 + 2y^2 + 3y - 5$	cubic $(ax^3 + bx^2 + cx + d)$	8	2	3	-5	–
$y^2 - y^3 + y^4 - 1$	$y^4 - y^3 + y^2 - 1$	quartic $(ax^4 + bx^3 + cx^2 + dx + e)$	1	-1	1	0	-1
$-2x^2$	$-2x^2$	quadratic $(ax^2 + bx + c)$	-2	0	0	–	–
$3 - x + 1$	$-x + 4$	linear $(ax + b)$	-1	4	–	–	–
$4x^2 - 5x - 5x^2$	$-x^2 - 5x$	quadratic $(ax^2 + bx + c)$	-1	-5	0	–	–
$\frac{2}{3}x$	$\frac{2}{3}x$	linear $(ax + b)$	$\frac{2}{3}$	0	–	–	–

7. (a) false (b) true (c) true

(d) false (e) false (f) true

(g) false (h) true (i) true

(j) true (k) true (l) false

(m) true (n) false (o) false

(p) false (q) false

Page 20

1. $= (7x^2 + 3x^2 - 5x^2) + (-x - 10x + x) + (2 - 4 + 8)$

   $= 5x^2 - 10x + 6$

2. $= (x^2 + 3x^2) + (3xy^2 - 5xy^2) + (5xy - 7xy) + (-2y^2 - 2y^2) + 4x^2y$

   $= 4x^2 - 2xy^2 - 2xy - 4y^2 + 4x^2y$

Page 21

1. $3x - 4y - z$ 2. $3n - 6k - 2$ 3. $-2a^2 - a - 5$

Page 25

1. $a + 2b - 4c$ 2. $6a - 2b - 4$ 3. $2x + y - 2$

Page 27

1. $(-2 \times 3)(a^2 \times a)(b) = -6a^3b$

2. $(-5 \times -3)(s \times s^2)(t \times t^2) = 15s^3t^3$

3. $(4 \times 2)(x \times x)(y \times y^2)(z) = 8x^2y^3z$

Page 28

1. $(-3)(a) + (-3)(-b) + (-3)(c) = -3a + 3b - 3c$

2. $4x(x) + 4x(y) = 4x^2 + 4xy$

3. $(-2b)(b^2) + (-2b)(-3bc) + (-2b)(c^2) = -2b^3 + 6b^2c - 2bc^2$

Page 29

1. $= 6m^2 + 10mn - 3mn - 5n^2$
   $= 6m^2 + 7mn - 5n^2$

2. $= (-3x)(5x) + 4(5x) + (-3x)(7) + (4)(7)$
   $= -15x^2 + 20x - 21x + 28$
   $= -15x^2 - x + 28$

3. $= a^2(a) + (-b)(a) + (a^2)(2b) + (-b)(2b)$
   $= a^3 - ab + 2a^2b - 2b^2$

4. $= (3z)(2z) + (-5)(2z) + (3z)(-3) + (-5)(-3)$
   $= 6z^2 - 10z - 9z + 15$
   $= 6z^2 - 19z + 15$

Page 30

1.
$$\begin{array}{rrrr}
 & 4x^2 & -\ 20x & +\ 25 \\
 & & 2x & -\ 5 \\ \hline
8x^3 & -\ 40x^2 & +\ 50x & \\
 & -\ 20x^2 & +\ 100x & -\ 125 \\ \hline
8x^3 & -\ 60x^2 & +\ 150x & -\ 125
\end{array}$$

2.
$$\begin{array}{rrrrrr}
a & -\ b & +\ c & & & \\
a & -\ b & +\ c & & & \\ \hline
a^2 & -\ ab & +\ ac & & & \\
 & -\ ab & & +\ b^2 & -\ bc & \\
 & & +\ ac & & -\ bc & +\ c^2 \\ \hline
a^2 & -\ 2ab & +\ 2ac & +\ b^2 & -\ 2bc & +\ c^2
\end{array}$$

Page 32

1. $6x^2$ Yes

2. $\frac{-8}{2} \times \frac{t^2}{t^3} = \frac{-4}{t}$ No

3. $\frac{-42}{-7} \times \frac{a^2}{a^3} \times \frac{b^2}{b} = \frac{6b}{a}$ No

4. $\frac{36}{8} \times \frac{x^3}{x^2} \times \frac{y^2}{y} \times z = \frac{9xyz}{2}$ Yes

Page 33

1. $= \frac{6x^2}{3x} + \frac{9x}{3x} = 2x + 3$ **Yes**

2. $= \frac{25y^2}{5y} + \frac{-15y^3}{5y} + \frac{10y^2}{5y} + \frac{-5y}{5y} = 5y^4 - 3y^2 + 2y - 1$ **Yes**

3. $= \frac{3y^2}{3y^2} + \frac{6y^3}{3y^2} + \frac{-2y}{3y^2} + \frac{-1}{3y^2} = 1 + 2y - \frac{2}{3y} - \frac{1}{3y^2}$ **No**

Page 35

$$\begin{array}{r}
x + 3 \\
2x - 1\,\overline{)\,2x^2 + 5x + 2} \\
\underline{2x^2 - x\phantom{{}+2}} \\
6x + 2 \\
\underline{6x - 3} \\
+5
\end{array}$$

No

$$\begin{array}{r}
x^2 - 2x + 2 \\
x + 2\,\overline{)\,x^3 + 0x^2 - 2x + 4} \\
\underline{x^3 + 2x^2\phantom{{}-2x+4}} \\
-2x^2 - 2x\phantom{{}+4} \\
\underline{-2x^2 - 4x}\phantom{{}+4} \\
2x + 4 \\
\underline{2x + 4} \\
0
\end{array}$$

Yes

$$\begin{array}{r}
x^2 - 3x + 3 \\
2x - 1\,\overline{)\,2x^3 - 7x^2 + 9x - 3} \\
\underline{2x^3 - x^2\phantom{{}+9x-3}} \\
-6x^2 + 9x\phantom{{}-3} \\
\underline{-6x^2 + 3x}\phantom{{}-3} \\
6x - 3 \\
\underline{6x - 3} \\
0
\end{array}$$

Yes

Pages 37-42

1. (a) add, additive
   (b) like, distributive
   (c) addition, subtraction, multiplication, division
   (d) $-4a + 5b$
   (e) additive inverse
   (f) zero
   (g) distributive
   (h) $-x + y + 2z$
   (i) descending
   (j) $18ab - 15b^2$

2. (c) $1 - (a + 3b + 2c)$
   (d) $1 - (-xy - x)$
   (e) $1 - (4ab - 2ac + 3ad)$
   (f) $1 - (-p + q - r)$

3. (a) $2x + 2x = 4x$
   (b) $5x^2 - 6x + 7$
   (c) $= 6x^2 - 3x + 2 - 7x^2 - 3x + 5$
   $= -x^2 - 6x + 7$
   (d) $= x^2 + 3x^2 - 2x^2 + 6xy - 9xy - 5y^2 + 6y^2 - y^2$
   $= 2x^2 - 3xy + 0$
   $= 2x^2 - 3xy$

(e) $= r^2 + r + 6 - 2r^2 + 2r + 5$

$= -r^2 + 3r + 11$

(f) $= 2x^2 + 3x - 5 + x^2 - 5x + 7 - 3x^2 - 2x - 1$

$= 0x^2 - 4x + 1$

$= -4x + 1$

(g) $= a - 2b - a - b + 2a - 5b + a + b$

$= 3a - 7b$

(h) $-3m^3n^3t^5$

(i) $-2m^3 + 6m^2n - 2mn^2$

(j) $-8x^4y + 12x^3y^3 - 4x^2y^5$

(k) $= a(a^2 + 3) - 5(a^2 + 3)$

$= a^3 + 3a - 5a^2 - 15$

(l) $= 3x(4x + 1) - 7(4x + 1)$

$= 12x^2 + 3x - 28x - 7$

$= 12x^2 - 25x - 7$

(m) $= 3a(2a - 5b) + b(2a - 5b)$

$= 6a^2 - 15ab + 2ab - 5b^2$

$= 6a^2 - 13ab - 5b^2$

(n) $= 2y^2(4y^2 - 5) + 3(4y^2 - 5)$

$= 8y^4 - 10y^2 + 12y^2 - 15$

$= 8y^4 + 2y^2 - 15$

(o) $-5m^3n^5$

(p) $\frac{5a}{a} + \frac{b}{a} = 5 + \frac{b}{a}$

(q) $a^2 + 3a - 6$

(r) $\frac{-12a^4}{x^4} + \frac{4}{x^3}$

(s) $-4n - m$

(t) $-4a^3 + 8a^2b - 3ab^2$

4. (b)

$$\begin{array}{r} c^2d + 3c^2d^2 - cd^2 \\ 2c^2d + 5c^2d^2 - 4cd^2 \\ \hline 3c^2d + 8c^2d^2 - 5cd^2 \end{array} \qquad \begin{array}{r} c^2d + 3c^2d^2 - cd^2 \\ 2c^2d + 5c^2d^2 - 4cd^2 \\ \hline -c^2d - 2c^2d^2 + 3cd^2 \end{array}$$

(c)

$$\begin{array}{r} t^3 - t^2 + 0t + 1 \\ -2t^3 + 4t^2 - 5t + 4 \\ \hline -t^3 + 3t^2 - 5t + 5 \end{array} \qquad \begin{array}{r} t^3 - t^2 + 0t + 1 \\ -2t^3 + 4t^2 - 5t + 4 \\ \hline 3t^3 - 5t^2 + 5t - 3 \end{array}$$

(d)

$$\begin{array}{r} 2y^4 - 3y^3 + y^2 - 4 \\ -3y^4 + 2y^3 + 5y^2 + 0 \\ \hline -y^4 - y^3 + 6y^2 - 4 \end{array} \qquad \begin{array}{r} 2y^4 - 3y^3 + y^2 - 4 \\ -3y^4 + 2y^3 + 5y^2 + 0 \\ \hline 5y^4 - 5y^3 - 4y^2 - 4 \end{array}$$

5. (a)

$$\begin{array}{r}
x^2 - 3x - 5 \\
x + 2 \\ \hline
x^3 - 3x^2 - 5x \qquad\quad \\
+ 2x^2 - 6x - 10 \\ \hline
x^3 - x^2 - 11x - 10
\end{array}$$

(b)

$$\begin{array}{l}
-a + b - c \\
\qquad\; a - b \\ \hline
-a^2 + ab - ac \\
\quad + ab \qquad\quad - b^2 + bc \\ \hline
-a^2 + 2ab - ac - b^2 + bc
\end{array}$$

(c)

$$\begin{array}{r}
x^2 - xy + y^2 \qquad \\
x + y \qquad \\ \hline
x^3 - x^2y + xy^2 \qquad \\
+ x^2y - xy^2 + y^3 \\ \hline
x^3 + y^3 \qquad\qquad\qquad
\end{array}$$

(d)

$$\begin{array}{l}
+x + y + z \\
-x + y - z \\ \hline
-x^2 - xy - xz \\
\quad + xy \qquad\quad + y^2 + yz \\
\qquad\quad - xz \qquad\quad - yz - z^2 \\ \hline
-x^2 - 2xz + y^2 - z^2
\end{array}$$

6. (a)

$$\begin{array}{r|r}
 & 4c^2 - 4c + 1 \\ \hline
2c - 1 & 8x^3 - 12c^2 + 6c - 1 \\
 & 8c^3 - 4c^2 \qquad\qquad\quad \\ \cline{2-2}
 & - 8c^2 + 6c \qquad \\
 & - 8c^2 + 4c \qquad \\ \cline{2-2}
 & 2c - 1 \\
 & 2c - 1 \\ \cline{2-2}
 & 0
\end{array}$$

(b)

$$\begin{array}{r|r}
 & 5x + 2 \\ \hline
5x - 2 & 25x^2 + 0x - 4 \\
 & 25x^2 - 10x \qquad \\ \cline{2-2}
 & 10x - 4 \\
 & 10x - 4 \\ \cline{2-2}
 & 0
\end{array}$$

(c)

$$\begin{array}{r|r}
 & y^2 + 2y - 2 \\ \hline
y - 2 & y^3 + 0y^2 - 6y + 7 \\
 & y^3 - 2y^2 \qquad\qquad\quad \\ \cline{2-2}
 & 2y^2 - 6y \qquad \\
 & 2y^2 - 4y \qquad \\ \cline{2-2}
 & - 2y + 7 \\
 & - 2y + 4 \\ \cline{2-2}
 & 3
\end{array}$$

(d)

$$\begin{array}{r|r}
 & 7a + 5b \\ \hline
4a + 3b & 28a^2 + 41ab + 15b^2 \\
 & 28a^2 + 21ab \qquad\quad \\ \cline{2-2}
 & 20ab + 15b^2 \\
 & 20ab + 15b^2 \\ \cline{2-2}
 & 0
\end{array}$$

7. (a) $= -8x^2 + 4x + 6x + 12x^2$

$= 4x^2 + 10x$

(b) $= 2xy + 4x - 15y + 6xy$

$= 4x - 15y + 8xy$

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Factors	Mental Calculations			Trinomial Product
	First Term	Second Term	Third Term	
$(x + 4y)(3x - 5y)$	$(x)(3x) = 3x^2$	$(4)(3x) + (x)(-5y)$ $= 12xy - 5xy$ $= 7xy$	$(4y)(-5y)$ $= -20y^2$	$3x^2 + 7xy - 20y^2$
$(x - 4)(x + 7)$	$(x)(x) = x^2$	$(-4)(x) + (x)(7)$ $= -4x + 7x$ $= 3x$	$(-4)(7)$ $= -28$	$x^2 + 3x - 28$
$(-x + 3)(2x - 4)$	$(-x)(2x) = -2x^2$	$-x(-4) + 3(2x)$ $= 4x + 6x$ $= 10x$	$(3)(-4)$ $= -12$	$-2x^2 + 10x - 12$
$(4a + 2b)(2a - 3b)$	$4a(2a) = 8a^2$	$4a(-3b) + 2b(2a)$ $= -12ab + 4ab$ $= -8ab$	$(2b)(-3b)$ $= -6b^2$	$8a^2 - 8ab - 6b^2$
$(6x - 1)(3x - 2)$	$(6x)(3x) = 18x^2$	$6x(-2) + (-1)(3x)$ $= -12x - 3x$ $= -15x$	$(-1)(-2)$ $= 2$	$18x^2 - 15x + 2$

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1. $6x^2 - x - 40$

2. $6y^2 + 7xy - 3x^2$

3. $-2a^2 - 6ab + 20b^2$

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Factors	Square of First Term	Square of Second Term	Binomial Product (Difference of Squares)
$(a - 7b)(a + 7b)$	$a(a) = a^2$	$(7b)^2 = 49b^2$	$a^2 - 49b^2$
$(5rs - t)(5rs + t)$	$(5rs)^2 = 25r^2s^2$	$(t)^2 = t^2$	$25r^2s^2 - t^2$
$(m^2n - 2p)(m^2n + 2p)$	$(m^2n)^2 = m^4n^2$	$(2p)^2 = 4p^2$	$m^4n^2 - 4p^2$
$(6ab + 8c^2)(6ab - 8c^2)$	$(6ab)^2 = 36a^2b^2$	$(8c^2)^2 = 64c^4$	$36a^2b^2 - 64c^4$
$(3m^3 + nt)(3m^3 - nt)$	$(3m^3)^2 = 9m^6$	$(nt)^2 = n^2t^2$	$9m^6 - n^2t^2$
$\left(\frac{1}{3}y + 4\right)\left(\frac{1}{3}y - 4\right)$	$\left(\frac{1}{3}y\right)^2 = \frac{1}{9}y^2$	$(4)^2 = 16$	$\frac{1}{9}y^2 - 16$
$\left(\frac{xy}{5} - 1\right)\left(\frac{xy}{5} + 1\right)$	$\left(\frac{xy}{5}\right)^2 = \frac{x^2y^2}{25}$	$(1)^2 = 1$	$\frac{x^2y^2}{25} - 1$

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Binomial Squared	Calculations			Expansion
	First Term	Middle Term	Last Term	
$(x + 2y)^2$	$(x)^2 = x^2$	$2(x)(2y)$ $= 4xy$	$(2y)^2 = 4y^2$	$x^2 + 4xy + 4y^2$
$(-2a + 3b)^2$	$(-2a)^2 = 4a^2$	$2(-2a)(3b)$ $= -12ab$	$(3b)^2 = 9b^2$	$4a^2 - 12ab + 9b^2$
$(ab - 5c)^2$	$(ab)^2 = a^2b^2$	$2ab(-5c)$ $= -10abc$	$(5c)^2 = 25c^2$	$a^2b^2 - 10abc + 25c^2$
$(3x + 8y)^2$	$(3x)^2 = 9x^2$	$2(3x)(8y)$ $= 48xy$	$(8y)^2 = 64y^2$	$9x^2 + 48xy + 64y^2$

Pages 52-54

1. (a) $a^2 - 16$

(b) $r^2 + 6r + 8$

(c) $t^2 - 11t + 30$

(d) $x^2 - 14x + 49$

(e) $9x^2 - y^2$

(f) $2x^2 - 3x - 2$

(g) $25x^2 - 70x + 49$

(h) $-2x^2 + 5x - 2$

(i) $15x^2 - 23xy + 4y^2$

(j) $4x^2 + 4bx + b^2$

(k) $4 - m^4$

(l) $6m^4 - m^2 - 15$

(m) $9 - 24a + 16a^2$

(n) $9a^2 - b^2c^4$

(o) $10x^4 - x^2 - 21$

(p) $x^2y^2 - 8xy + 16$

(q) $9m^2n^2 - 4$

(r) $15x^2 - 13x + 2$

(s) $4y^4 + 16y^2z + 16z^2$

(t) $4a^4 - b^4$

(u) $9m^2 - 18mn - 7n^2$

(v) $16y^2 + 56y + 49$

2. (a) 3

(b) 2

(c) 4

(d) 3

(e) 3

(f) 2

(g) 3

(h) 4

(i) 2

(j) 2

(k) 3

(l) 4

(m) 3

3. (a) $= x^2 + 6x + 9 + 9x^2 - 6x + 1$

$= 10x^2 + 10$

(b) $(x^2 - 2x + 1) - (-x^2 + 9)$

$= x^2 - 2x + 1 + x^2 - 9$

$= 2x^2 - 2x - 8$

(c) $-5x(x^2 + 6xy + 9y^2)$

$= -5x^3 - 30x^2y - 45xy^2$

(d) $(x^2 - y^2) + (x^2 - 4xy + 4y^2)$

$= x^2 - y^2 + x^2 - 4xy + 4y^2$

$= 2x^2 - 4xy + 3y^2$

(e) $(2a^2 + 2a) - (a^2 + 2a - 3)$

$= 2a^2 + 2a + (-a^2 - 2a + 3)$

$= a^2 + 3$

(f) $(x + 1) + (2x - 2) - (x^2 + 6x + 9)$

$= (x + 1) + (2x - 2) + (-x^2 - 6x - 9)$

$= -x^2 - 3x - 10$

(g) $(16a^2 - 4b^2) + (9b^2 - 12ab + 4a^2)$

$= 20a^2 - 12ab + 5b^2$

END OF LESSON D

LESSON E

Page 5

1. $5(4x + 3)$
2. $2x(x) + 2x(2y) = 2x(x + 2y)$
3. $6abc(2c) + 6abc(-a) = 6abc(2c - a)$
4. $3b(1) + 3b(2c) + 3b(-5b) = 3b(1 + 2c - 5b)$

Page 8

1. $y^2 - 4^2 = (y - 4)(y + 4)$
2. $(3t)^2 - (5r)^2 = (3t - 5r)(3t + 5r)$
3. $(6t)^2 - 1^2 = (6t - 1)(6t + 1)$
4. $(m^3)^2 - n^2 = (m^3 - n)(m^3 + n)$
5. $(8y)^2 - (7x)^2 = (8y - 7x)(8y + 7x)$
6. $4a^2b^2 - \frac{1}{9} = (2ab)^2 - \left(\frac{1}{3}\right)^2 = \left(2ab - \frac{1}{3}\right)\left(2ab + \frac{1}{3}\right)$

Page 9

$$
\begin{aligned}
& (3mx)(4x^2) + (3mx)(-25y^2) \\
= \; & 3mx(4x^2 - 25y^2) \\
= \; & 3mx[(2x)^2 - (5y)^2] \\
= \; & 3mx(2x - 5y)(2x + 5y)
\end{aligned}
$$

Page 12

1. $(5a - b)^2$
2. $x^2 + 2(x)(7) + (7)^2 = (x + 7)^2$
3. $(3x)^2 + 2(3x)(-4) + (-4)^2 = (3x - 4)^2$
4. $x^2 + 2(x)(-3y) + (-3y)^2 = (x - 3y)^2$
5. $(x^2)^2 + 2(x^2)(6y^2) + (6y^2)^2 = (x^2 + 6y^2)^2$

$$\begin{aligned} 45a^3b + 30a^2b^2 + 5ab^3 &= (5ab)(9a^2) + (5ab)(6ab) + (5ab)(b^2) \\ &= 5ab(9a^2 + 6ab + b^2) \\ &= 5ab[(3a)^2 + 2(3a)(b) + (b)^2] \\ &= 5ab(3a + b)^2 \end{aligned}$$

Page 16

1. $(y - 4)(y + 3)$
2. $(a - 3)(a - 1)$
3. $(x + 5y)(x - 2y)$
4. $(x + 1)(x - 7)$
5. $(x - 8)(x + 4)$
6. $(m + 7)(m + 2)$
7. $(x - 6)(x - 5)$
8. $(a - 6b)(a + 2b)$

Page 17

2. $(t + 5)(t - 3)$ since 5 and -3
3. $(r + 4)(r - 7)$ since 4 and -7
4. $(y - 4)(y - 5)$ since -4 and -5
5. $(x + 2)(x - 7)$ since 2 and -7
6. $(x - 6)(x - 3)$ since -6 and -3
7. $(t - 11)(t + 2)$ ssince -11 and 2
8. $(x - 5)(x + 7)$ since -5 and 7

$$\begin{aligned} 3x^2 + 9xy - 54y &= (3y)(x^2) + (3y)(3x) + (3y)(-18) \\ &= 3y(x^2 + 3x - 18) \\ &= 3y(x + 6)(x - 3) \end{aligned}$$

Page 21

$(2x + 1)(x - 14)$ CPS is $-2yx$.

$(2x - 14)(x + 1)$ CPS is $-12x$.

$(2x - 1)(x + 14)$ CPS is $27x$.

$(2x + 14)(x - 1)$ CPS is $12x$.

$(12x - 1(x - 1)$ CPS is $-27x$

$(2x - 2)(x + 7)$ CPS is $12x$.

$(2x + 7)(x - 2)$ CPS is $3x$.

$(2x + 2)(x - 7)$ CPS is $-12x$.

$(2x - 7)(x + 2)$ CPS is $-3x$.

$(4x - 3)(3x - 5)$ CPS is $-29x$.

$(12x - 1)(x - 15)$ CPS is **$-181x$**.

$(12x - 3)(x - 5)$ CPS is **$-63x$**.

$(12x - 5)(x - 3)$ CPS is **$-41x$**.

$(4x - 15)(3x - 1)$ CPS is **$-49x$**.

$(4x - 1)(3x - 15)$ CPS is **$-63x$**.

$(4x - 5)(3x - 3)$ CPS is **$-27x$**.

$(6x - 15)(2x - 1)$ CPS is **$-36x$**.

$(6x - 1)(2x - 15)$ CPS is **$-92x$**.

$(6x - 3)(2x - 5)$ CPS is **$-36x$**.

$(6x - 5)(2x - 3)$ CPS is **$-28x$**.

Page 22

1. $2x^2 + 7x + 3$

   (a) Since the first term of the trinomial is **$2x^2$**, the first terms of the binomial factors must be **$2x$** and **x**.

   $$(2x \qquad)(x \qquad)$$

   (b) Since the last term of the trinomial is **3**, the second terms of the binomial factors must be two integers with like signs giving a product of **3**. Since the middle term of the trinomial is positive, the two integers must be **positive**. Thus, the second terms of the binomial factos must be **3** and **1**.

   $$(\qquad 3)(\qquad 1)$$

   (c) The pair of binomials that gives a CPS of $7x$ is: **$(2x + 1)(x + 3)$**

2. $4x^2 - 8x - 21$

   (a) Since the first term of the trinomial is **$4x^2$**, the first terms of the binomial factors must be **$4x$** and x or **$2x$** and **$2x$**.

   $$\begin{array}{l}(4x \qquad)(x \qquad)\\(2x \qquad)(2x \qquad)\end{array}$$

   (b) Since the last term of the trinomial is **-21**, the second terms of the binomial factors must be two integers with **unlike** signs that give a product of **-21**. Thus, the second terms must be 1 and -21, -1 and **21**, 3 and **-7**, or -3 and 7.

   $$\begin{array}{l}(\qquad + 1)(\qquad - 21)\\(\qquad - 1)(\qquad + 21)\\(\qquad + 3)(\qquad - 7)\\(\qquad - 3)(\qquad + 7)\end{array}$$

   (c) Test possible combinations. Stop when you obtain a CPS of $-8x$.

Possible Factors	**CPS**
$(4x + 1)(x - 21)$	$-83x$
$(2x + 3)(2x - 7)$	$-8x$

Thus, $4x^2 - 8x - 21 = (2x + 3)(2x - 7)$.

3. $3x^2 + x - 30$

(a) Since the first term of the trinomial is $\mathbf{3x^2}$, the first terms of the binomial factors must be **3x** and **x**.

$$(3x \quad)(x \quad)$$

(b) Since the last term of the trinomial is **−30**, the second terms of the binomial factors must be two integers with **unlike** signs that give a product of **−30**. Thus, the second terms must be 1 and −30, −1 and **30**, 3 and **−10**, −3 and **10**, 5 and **−6**, or −5 and **6**, −2 and 15, or 2 and −15.

$(\quad +1)(\quad -30)$
$(\quad -1)(\quad +30)$
$(\quad +3)(\quad -10)$
$(\quad -3)(\quad +10)$
$(\quad +5)(\quad -6)$
$(\quad -5)(\quad +6)$
$(\quad -2)(\quad +15)$
$(\quad +2)(\quad -15)$

(c) Test possible combinations. Stop when you obtain a CPS of x.

Possible Factors	CPS
$(3x + 1)(x - 30)$	$-89x$
$\mathbf{(3x + 10)(x - 3)}$	$\mathbf{x}$

Thus, $3x^2 + x - 30 = \mathbf{(3x + 10)(x - 3)}$.

Page 24

1. $12x^2 + 7x - 10 = (3x - 2)(4x + 5)$
2. $24x^2 - 38x + 15 = (4x - 3)(6x - 5)$
3. $2y^2 + 13y + 15 = (2y + 3)(y + 5)$
4. $8x^2 + 10x - 3 = (4x - 1)(2x + 3)$
5. $10x^2 - 29x + 10 = (5x - 2)(2x - 5)$

1. $5x^2 - 14x + 8 = (5x - 4)(x - 2)$
2. $3x^2 + 7x - 6 = (3x - 2)(x + 3)$
3. $15x^2 - 2x - 24 = (5x + 6)(3x - 4)$
4. $12x^2 - 11x - 15 = (4x + 3)(3x - 5)$
5. $35x^2 + 13x - 12 = (7x - 3)(5x + 4)$

Page 25

$$\begin{aligned} 4x^2y + 10xy - 24y &= (2y)(2x^2) + (2y)(5x) + (2y)(-12) \\ &= 2y(2x^2 + 5x - 12) \\ &= 2y(2x - 3)(x + 4) \end{aligned}$$

Pages 27-31

1. (a) 4 (b) sum, product (c) prime

(d) $5x^2y$ (e) quadratic (f) 6, 5; 5, 1

(g) $40x$ (h) $-7x$ (i) difference, squares

2. (a) true (b) false (c) false

(d) true (e) false (f) true

(g) false (h) false (i) false

(j) false (k) false (l) false

(m) true (n) false (o) true

3. (a) $9a^4b^3(5b^2 - 2a^2)$

(b) $3(x) + 3(-2) = 3(x - 2)$

(c) $x(x) - x(y) = x(x - y)$

(d) $a(a) + a(-b) + a(-1) = a(a - b - 1)$

(e) $6r^3(1) + 6r^3(2r) + 6r^3(4r^2) = 6r^3(1 + 2r + 4r^2)$

(f) $7y^2(2y) + 7y^2(-3) + 7y^2(7y^4) = 7y^2(2y - 3 + 7y^4)$

(g) $16a(5a) + 16a(-b) = 16a(5a - b)$

4. (a) $(p - q)(p + q)$

(b) $= 2a(a^2 - 25b^2)$
$= 2a(a - 5b)(a + 5b)$

(c) $= (5x)^2 - (6y)^2$
$= (5x - 6y)(5x + 6y)$

(d) $= 10x(x^2 - 9)$
$= 10x(x - 3)(x + 3)$

(e) $= (a^3)^2 - (3b^2)^2$
$= (a^3 - 3b^2)(a^3 + 3b^2)$

(f) $= (x + y - z)(x + y + z)$

(g) $= 3ax(9x^4 - a^2)$
$= 3ax(3x^2 - a)(3x^2 + a)$

(h) $= [m - (n + 2)][m + (n + 2)]$
$= (m - n - 2)(m + n + 2)$

5. (a) $(2n + 7)^2$

(b) $(3ab - 5)^2$

(c) $(5x - 9y)^2$

(d) $(3x^2 + 1)^2$

6. (a) $2(2x^2 + 5x + 2)$
$= 2(2x + 1)(x + 2)$

(b) $(x - 9)(x + 3)$

(c) $(x - 11)(x - 4)$

(d) $3(x^2 + 3x - 28)$
$= 3(x + 7)(x - 4)$

(e) $(4y - 3)(3y + 1)$

(f) $(3a - 2)(a - 1)$

(g) $(2x - 3)(x + 1)$

(h) $(4a + b)(a + 3b)$

(i) $-(x + 5)(x - 2)$

(j) $-(6x^2 - x - 2)$
$= -(3x - 2)(2x + 1)$

(k) $= 10(3x^2 + x - 10)$
$= 10(3x - 5)(x + 2)$

(l) $= m(m^2 + 5m - 36)$
$= m(m + 9)(m - 4)$

(m) $= (3a + 4b)(a - 9b)$

(n) $= (5x^2 + 2)(2x^2 - 1)$

7. (a) $= (x - 8)(x + 7)$

(b) $= x(x^2 - 1)$
$= x(x - 1)(x + 1)$

(c) $= (a^2 - 1)(a^2 + 1)$
$= (a - 1)(a + 1)(a^2 + 1)$

(d) $= 2(x^2 - 6x + 9)$
$= 2(x - 3)^2$

(e) $= (4x + 1)(x - 1)$

(f) $= 3x(9x^2 - 1)$
$= 3x(3x - 1)(3x + 1)$

(g) $= (10b^2 - 1)^2$

(h) $= 10(2a^2 + ab - 3b^2)$
$= 10(2a + 3b)(a - b)$

(i) $= xy(1 - x^2y^2)$

$= xy(1 - xy)(1 + xy)$

(j) $= a(x^2 + 4x - 5)$

$= a(x + 5)(x - 1)$

(k) $2y^2(8y^2 - 6y - 9)$

$= 2y^2(4y + 3)(2y - 3)$

(l) $y^2(10x^2 + 7x - 12)$

$= y^2(5x - 4)(2x + 3)$

(m) $(16x + 1)(x - 1)$

(n) $5(1 - 16a^2)$

$= 5(1 - 4a)(1 + 4a)$

(o) $ab(a^2b^2 - 2ab - 3)$

$= ab(ab + 1)(ab - 3)$

(p) $-(5y^2 + 6y - 11)$

$= -(5y + 11)(y - 1)$

(q) $a(a^4 - 81)$

$= a(a^2 - 9)(a^2 + 9)$

$= a(a - 3)(a + 3)(a^2 + 9)$

(r) $(x + y - 1)(x + y + 1)$

(s) $3uv^2(2u - 3v + 1)$

(t) $(2x - 3)^2$

(u) $(8y - 7x)(8y + 7x)$

(v) $(4y + 3)(3y - 1)$

Page 35

Rational Expression	Common Factors	Cancellation	Lowest Terms
$\frac{-18a^2b^2}{24ab^4}$	6, a, b^2	$\frac{-\overset{-3}{\cancel{18}}\overset{a}{\cancel{a^2}}\overset{1}{\cancel{b^2}}}{\underset{4}{\cancel{24}}\underset{1}{\cancel{a}}\underset{b^2}{\cancel{b^4}}}$	$\frac{-3a}{4b^2}$
$\frac{7x^2}{21xy}$	7, x	$\frac{\overset{1}{\cancel{7}}\overset{x}{\cancel{x^2}}}{\underset{3}{\cancel{21}}\underset{1}{\cancel{x}}y}$	$\frac{x}{3y}$
$\frac{-4ab}{16a^2b^3}$	4, a, b	$\frac{\overset{-1}{\cancel{-4}}\overset{1}{\cancel{a}}\overset{1}{\cancel{b}}}{\underset{4}{\cancel{16}}\underset{a}{\cancel{a^2}}\underset{b^2}{\cancel{b^3}}}$	$\frac{-1}{4ab^2}$
$\frac{64y^4z^5}{16y^3z^2}$	16, y^3, z^2	$\frac{\overset{4}{\cancel{64}}\overset{y}{\cancel{y^4}}\overset{z^3}{\cancel{z^5}}}{\underset{1}{\cancel{16}}\underset{1}{\cancel{y^3}}\underset{1}{\cancel{z^2}}}$	$4yz^3$
$\frac{-9ab^2c^2}{-3ab^2c^3}$	-3, a, b^2, c^2	$\frac{-\overset{3}{\cancel{9}}\overset{1}{\cancel{a}}\overset{1}{\cancel{b^2}}\overset{1}{\cancel{c^2}}}{-\underset{1}{\cancel{3}}\cancel{a}\cancel{b^2}\underset{c}{\cancel{c^3}}}$	$\frac{3}{c}$

Page 36

1. $\dfrac{x-2}{2(x-5)}$

2. $\dfrac{3}{6(x-2)} = \dfrac{1}{2(x-2)}$

3. $\dfrac{(y-2)(y+1)}{(2y-1)(y-2)} = \dfrac{y+1}{2y-1}$

4. $\dfrac{(x-1)(x+1)}{(x-2)(x-1)} = \dfrac{x+1}{x-2}$

5. $\dfrac{3(x^2-y^2)}{(x+y)(x+y)} = \dfrac{3(x-y)(x+y)}{(x+y)(x+y)} = \dfrac{3(x-y)}{x+y}$

Page 38

1. $\dfrac{-1}{3c-2d}$

2. $\dfrac{(\overset{-1}{\cancel{x-3}})(x-2)}{(\underset{1}{\cancel{3-x}})(3+x)} = \dfrac{-1(x-2)}{3+x}$

3. $\dfrac{(\overset{-1}{\cancel{c-d}})(3)(\overset{1}{\cancel{t+c}})}{2(\underset{1}{\cancel{c+t}})(\underset{1}{\cancel{d-c}})} = \dfrac{-3}{2}$

4. $\dfrac{(\overset{-1}{\cancel{2x-3}})(x+4)}{5(\underset{1}{\cancel{3-2x}})} = \dfrac{-1(x+4)}{5}$

Page 40

1. $\dfrac{-x}{2}$

2. $\dfrac{-5r \times 2rs}{s \times -20} = \dfrac{r^2}{2}$

3. $\dfrac{6x^4y \times 21a^2b^2}{7a^2b \times 8xy} = \dfrac{9x^3b}{4}$

4. $\dfrac{15t \times 3c}{20t^3} = \dfrac{9c}{4t^2}$

5. $\dfrac{-mc}{6a^2} \times \dfrac{-4a^2}{m^2pc} = \dfrac{2}{3amp}$

$$\frac{(x^2-5x-14)(x^2-9)}{(x^2+6x+9)(3x-21)} = \frac{(x-7)(x+2)(x-3)(x+3)}{(x+3)(x+3)(3)(x-7)} = \frac{(x+2)(x-3)}{3(x+3)}$$

Page 44

1. $\dfrac{2a \times 1}{1 \times 3} = \dfrac{2a}{3}$

2. $\dfrac{-5x^2y^3}{z} \times \dfrac{z^4}{15xy^4} = \dfrac{-xz^3}{3y}$

3. $\dfrac{xy}{6z^2} \times \dfrac{14z}{y^3} = \dfrac{7x}{3y^2z}$

4. $\dfrac{5a^2b}{c^3} \times \dfrac{1}{10a^2b} = \dfrac{1}{2c^3}$

$$\frac{a^2+4a-12}{a^2+9a-18} \times \frac{6a+18}{3a+12} = \frac{(a^2+4a-12)(6a+18)}{(a^2+9a+18)(3a+12)} = \frac{(a-2)(a+6)6(a+3)}{(a+6)(a+3)3(a+4)}$$

$$= \frac{2(a-2)}{(a+4)}$$

Page 47

1. $\frac{2-(n+2)}{m+n} = \frac{2+(-n-2)}{m+n} = \frac{-n}{m+n}$

2. $\frac{3-a+(a-b)}{b} = \frac{3-b}{b}$

3. $\frac{2x-8}{x^2-16} = \frac{2(x-4)}{(x-4)(x+4)} = \frac{2}{x+4}$

4. $\frac{(x+3)-(2x-4)}{x} = \frac{(x+3)+(-2x+4)}{x} = \frac{-x+7}{x}$

Page 49

1. $\frac{c+2a}{abc}$

2. $\frac{2y-5-3x}{xy}$

3. $\frac{2xz+3yz+4x^2}{x^2yz}$

4. $\frac{5+t}{r \times t} - \frac{3 \times r}{t \times r} = \frac{5t-3r}{rt}$

5. $\frac{4-(x-3)}{2x} = \frac{4+(-x+3)}{2x} = \frac{-x+7}{2x}$

Page 51

$\frac{x+y}{4xy} - \frac{y}{6x(x-y)} + \frac{x}{8y(x-y)}$ The L.C.D. is $24xy(x-y)$.

$$= \frac{(x+y)6(x-y)}{4xy6(x-y)} - \frac{y(4y)}{6x(x-y)4y} + \frac{x(3x)}{8y(x-y)(3x)}$$

$$= \frac{6(x+y)(x-y) - y(4y) + x(3x)}{24xy(x-y)}$$

$$= \frac{6(x^2-y^2) - 4y^2 + 3x^2}{24xy(x-y)}$$

$$= \frac{6x^2 - 6y^2 - 4y^2 + 3x^2}{24xy(x-y)}$$

$$= \frac{9x^2 - 10y^2}{24xy(x-y)}$$

Page 52

$$\frac{(2x+1)}{x(x-1)} - \frac{-x+5}{x-1}$$

$$= \frac{2x+1}{x(x-1)} - \frac{(-x+5)x}{(x-1)x}$$

$$= \frac{(2x+1)-(-x+5)x}{x(x-1)}$$

$$= \frac{(2x+1)-(-x^2+5x)}{x(x-1)}$$

$$= \frac{2x+1+x^2-5x}{x(x-1)}$$

$$= \frac{x^2-3x+1}{x(x-1)}$$

Pages 55-61 **Exercise – Rational Expressions**

1. (a) zero (b) (a − 2)

(c) quotient, polynomials (d) additive

(e) $\frac{1}{y}$ (f) numerators, denominators, $\frac{a-b}{c}$

(g) 1 (h) factors

(i) cancel (j) additive inverses

2. (a) $\frac{a(a+b)}{12a}$ (b) $\frac{6ab}{20ab}$ (c) $\frac{ab}{b^2}$

(d) $\frac{18k^2}{30k}$ (e) $\frac{x^2y}{y}$ (f) $\frac{5(x+y)}{(x+y)^2}$

3. (a) $a = 0$ (b) $a = 3$ (c) $x = 5$ or -5

(d) $x = 0$ or -2 (e) $a = -5$ or 2 (f) $x = 2$ or 3

4. (a) $\frac{13a^2bc}{26ab^3} = \frac{ac}{2b^2}$ (b) $\frac{c\overset{1}{\cancel{(y+z)}}}{2\underset{1}{\cancel{(y+z)}}} = \frac{c}{2}$ (c) in lowest terms

(d) in lowest terms (e) $\frac{\overset{1}{\cancel{5}}x\overset{y}{\cancel{y^2}}}{\underset{2}{\cancel{10}}\underset{1}{\cancel{y}}} = \frac{xy}{2}$ (f) $\frac{\overset{1}{\cancel{5}}(r+s)}{\underset{1}{\cancel{5}}rs} = \frac{r+s}{rs}$

(g) in lowest terms

(h) $\dfrac{\cancel{(2a-3b)}}{-1\cancel{(2a-3b)}} = \dfrac{1}{-1} = -1$

(i) $\dfrac{\cancel{(3x+2)}(3x+2)}{(3x-2)\cancel{(3x+2)}} = \dfrac{3x+2}{3x-2}$

(j) $\dfrac{(x+y)\cancel{(x+y)}}{\cancel{(x+y)}} = x+y$

(k) in lowest terms

(l) $\dfrac{\cancel{(4n-1)}(2n+3)}{\cancel{(4n-1)}(2n-3)} = \dfrac{2n+3}{2n-3}$

(m) $\dfrac{\cancel{(x-3)}(x+1)}{(x+5)\cancel{(x-3)}} = \dfrac{x+1}{x+5}$

(n) in lowest terms

(o) $\dfrac{\cancel{(r-3)}(r-3)}{-1\cancel{(r-3)}(r+3)} = \dfrac{-1(r-3)}{r+3}$

(p) in lowest terms

5. (a) false
(b) false
(c) false
(d) true
(e) false
(f) true
(g) true
(h) false
(i) false
(j) true
(k) false
(l) false
(m) true
(n) true
(o) false
(p) false
(q) true

6. (a) $\dfrac{2z^4}{15x^3y^2} \times \dfrac{5y}{8x} = \dfrac{z^4}{12x^4y}$

(b) $\dfrac{7ab}{5cd} \times \dfrac{15c^2d^2}{28a^2b^2} = \dfrac{3cd}{4ab}$

(c) $\dfrac{4a^2}{7} \times \dfrac{1}{8a} = \dfrac{a}{14}$

(d) $-15x^2y \times \dfrac{1}{3x^3y^2} = \dfrac{-5}{xy}$

(e) $\dfrac{(a-2)(a+3)}{3} \times \dfrac{1}{(a-2)(a+2)} = \dfrac{a+3}{3(a+2)}$

(f) $\dfrac{x(2x-1)}{(2x-1)(2x-1)} \times \dfrac{4(2x-1)}{x^2} = \dfrac{4}{x}$

(g) $\dfrac{(m+5n)(m-5n)}{(m-5n)(m-5n)} \times \dfrac{2(m-5n)}{(m+5n)} = \dfrac{2}{1} = 2$

(h) $\dfrac{(3n+1)(n+2)}{-1(3n-1)(3n+1)} \times \dfrac{(3n-1)(n-2)}{-1(n-2)(n+2)} = \dfrac{1}{(-1)(-1)} = \dfrac{1}{1} = 1$

(i) $\dfrac{r(r-s)(r+s)}{r^2s} \times \dfrac{3s(r+s)}{r(r-s)} = \dfrac{3(r+s)(r+s)}{r^2}$

(j) $\dfrac{(2x+1)(x+3)}{(x+3)(x-3)} \times \dfrac{x(x-3)}{(2x+1)(x+5)} = \dfrac{x}{x+5}$

(k) $$\frac{c + d - c + d}{c} = \frac{2d}{c}$$

(l) $$\frac{r - 2 + 3r + 2 - r - 5}{2r} = \frac{3r - 5}{2r}$$

(m) $$\frac{2(bc) + 3(ac) - (ab)}{abc} = \frac{2bc + 3ac - ab}{abc}$$

(n) $$\frac{4x}{(x - 6)(x + 6)} + \frac{3(x + 6)}{(x - 6)(x + 6)} = \frac{4x + 3x + 18}{(x - 6)(x + 6)} = \frac{7x + 18}{(x - 6)(x + 6)}$$

(o) $$\frac{a(a - b) + b(a + b) - 1}{(a + b)(a - b)} = \frac{a^2 - ab + ab + b^2 - 1}{a^2 - b^2} = \frac{a^2 + b^2 - 1}{a^2 - b^2}$$

(p) L.C.D $= (y + 2)(y + 4)$

$$\frac{(y + 1)(y + 4)}{(y + 2)(y + 4)} - \frac{2(y + 2)}{(y + 4)(y + 2)} + \frac{4}{(y + 4)(y + 2)}$$

$$= \frac{y^2 + 4y + 1y + 4 - 2y - 4 + 4}{(y + 2)(y + 4)}$$

$$= \frac{y^2 + 3y + 4}{(y + 2)(y + 4)}$$

(q) $$\frac{(x + 2)(2)(x + 5)}{(x - 5)(x + 5)(2)(x + 5)} - \frac{1(x + 5)(x - 5)}{2(x + 5)(x + 5)(x - 5)} + \frac{(x - 1)(2)(x - 5)}{(x + 5)(x + 5)(2)(x - 5)}$$

$$= \frac{2(x^2 + 7x + 10) - 1(x^2 - 25) + 2(x^2 - 6x + 5)}{2(x - 5)(x + 5)(x + 5)}$$

$$= \frac{2x^2 + 14x + 20 - x^2 + 25 + 2x^2 - 12x + 10}{2(x - 5)(x + 5)(x + 5)}$$

$$= \frac{3x^2 + 2x + 55}{2(x - 5)(x + 5)(x + 5)}$$

END OF LESSON E

LESSON F

Pages 10-11

1. $3x = 3$
   $x = 1$

2. $-6x + 12 = 0$
   $-6x = -12$
   $x = 2$

3. $-x = 7$
   $x = -7$

4. $4x + 8 = 16$
   $4x = 8$
   $x = 2$

Page 12

1. $3x + 3 - 2 = -4x - 6 + 5$
   $3x + 1 = -4x - 1$
   $7x + 1 = -1$
   $7x = -2$
   $x = \frac{-2}{7}$

2. $9x + 4 = 5x - 8$
   $4x + 4 = -8$
   $4x = -12$
   $x = -3$

Page 13

1. $8m = 240 + 5m$
   $3m = 240$
   $m = 80$

2. $4w + 10 - 10w - 3 = +16w + 8$
   $-6w + 7 = 16w + 8$
   $-22w + 7 = 8$
   $-22w = 1$
   $w = \frac{-1}{22}$

Pages 16-19 **Exercise – Linear Equations in One Variable**

1. (a) equivalent
   (b) indentity
   (c) -3, $\{-3\}$
   (d) dividing, -2
   (e) quadratic, two
   (f) least common denominator

2. (a) $4y + 82 = 58$
   $4y = -24$
   $y = -6$

   The solution set is $\{-6\}$

$3y + 82$	$58 - y$
$3(-6) + 82$	$58 - (-6)$
$-18 + 82$	$58 + 6$
64	64

(b)

$$\begin{aligned} 2x - 13 &= 3x + 7 \\ -x - 13 &= 7 \\ -x &= 20 \\ x &= -20 \end{aligned}$$

$2x - 13$	$3x + 7$
$2(-20) - 13$	$3(-20) + 7$
$-40 - 13$	$-60 + 7$
-53	-53

The solution set is $\{-20\}$.

(c)

$$\begin{aligned} 2x - 15 &= 5x \\ -3x - 15 &= 0 \\ -3x &= 15 \\ x &= -5 \end{aligned}$$

$2x - 15$	$5x$
$2(-5) - 15$	$3(-5)$
$-10 - 15$	-25
-25	

The solution set is $\{-5\}$.

(d)

$$\begin{aligned} \frac{x}{4} + 7 &= -3 \\ \frac{x}{4} &= -10 \\ x &= -40 \end{aligned}$$

$\frac{4}{x} + 7$	-3
$\frac{-40}{4} + 7$	-3
$-10 + 7$	
-3	

The solution set is $\{-40\}$.

(e)

$$\begin{aligned} 3x - 2 + 4x + 7 &= 8x - 6 \\ 7x + 5 &= 8x - 6 \\ -x + 5 &= -6 \\ -x &= -11 \\ x &= 11 \end{aligned}$$

$3x - 2 + 4x + 7$	$8x - 6$
$3(11) - 2 + 4(11) + 7$	$8(11) - 6$
$33 - 2 + 44 + 7$	$88 - 6$
82	82

The solution set is $\{11\}$.

(f)

$$3x - (4 + 5x) = 5(2x + 1)$$
$$3x - 4 - 5x = 10x + 5$$
$$-2x - 4 = 10x + 5$$
$$-12x - 4 = 5$$
$$-12x = 9$$
$$x = \frac{-9}{12}$$
$$x = \frac{-3}{4}$$

The solution set is $\left\{\frac{-3}{4}\right\}$.

$3x - (4 + 5x)$	$5(2x + 1)$
$3\left(\frac{-3}{4}\right) - \left[4 + 5\left(\frac{-3}{4}\right)\right]$	$5\left[2\left(\frac{-3}{4}\right) + 1\right]$
$\frac{-9}{4} - \left(\frac{16 - 15}{4}\right)$	$5\left(\frac{-6}{4} + 1\right)$
$\frac{-9}{4} - \frac{1}{4}$	$5\left(\frac{-6 + 4}{4}\right)$
$\frac{-10}{4}$	$5\left(\frac{-2}{4}\right)$
$\frac{-5}{2}$	$\frac{-5}{2}$

(g)

$$\frac{x + 2}{2} = \frac{x - 1}{3}$$
$$\frac{6(x + 2)}{2} = \frac{(x - 1)6}{3}$$
$$3x + 6 = 2x - 2$$
$$x + 6 = -2$$
$$x = -8$$

The solution set is $\{-8\}$.

$\frac{x + 2}{2}$	$\frac{x - 1}{3}$
$\frac{-8 + 2}{2}$	$\frac{-8 - 1}{3}$
$\frac{-6}{2}$	$\frac{-9}{3}$
-3	-3

(h)

$$\frac{10(2x - 3)}{5} - 7(10) = \frac{(5x + 1)10}{10}$$
$$4x - 6 - 70 = 5x + 1$$
$$4x - 76 = 5x + 1$$
$$-x - 76 = 1$$
$$-x = 77$$
$$x = -77$$

The solution set is $\{-77\}$.

$\frac{2x - 3}{5} - 7$	$\frac{5x + 1}{10}$
$\frac{2(-77) - 3}{5} - 7$	$\frac{5(-77) + 1}{10}$
$\frac{-154 - 3}{3} - \frac{35}{5}$	$\frac{-385 + 1}{10}$
$\frac{-157 - 35}{5}$	$\frac{-384}{10}$
$\frac{-192}{5}$	$\frac{-192}{5}$

(i) $$12\left(\frac{x}{3}\right) - \frac{12(x+5)}{4} = \frac{12(2x+3)}{6}$$

$$4x - 3x - 15 = 2(2x+3)$$
$$x - 15 = 4x + 6$$
$$-3x - 15 = 6$$
$$-3x = 21$$
$$x = -7$$

The solution set is $\{-7\}$.

$\frac{x}{3} - \frac{x+5}{4}$	$\frac{2x+3}{6}$
$\frac{-7}{3} - \frac{-7+5}{4}$	$\frac{2(-7)+3}{6}$
$\frac{-7}{3} - \frac{-2}{4}$	$\frac{-14+3}{6}$
$\frac{-7}{3} + \frac{1}{2}$	$\frac{-11}{6}$
$\frac{-14}{6} + \frac{3}{6}$	
$\frac{-11}{6}$	

3. Column II

1. E 2. G 3.E

4. B 5. A 6. I

7. A 8. D 9. E

10. C 11. D 12. H

13. F 14. E 15. B

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1 (a) true (b) false (c) true

(d) true (d) true (f) false

2. (a) false (b) true (c) false (d) true

Pages 30-32 **Exercise – Linear Inequalities in One Variable**

1. (a) multiply, -3, direction (b) equivalent, solution

(c) less, 10 (d) multiply, divide, negative

(e) $\{x \mid x \leq -4\}$

2. (a) Subtract 2 from both sides.

(b) Add 5 to both sides.

Multiply both sides by 4.

(c) Add 3 to both sides.

Multiply both sides by -1 and change arrow.

(d) Subtract 2x from both sides.

Subtract 3 from both sides.

Divide both sides by -7 and change arrow.

(e) Subtract 7 from both sides.

Multiply both sides by $\frac{-2}{3}$ and change arrow.

(f) Multiply each term by 12.

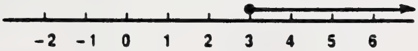
Subtract 3x from both sides.

Divide both sides by 5.

3. (a)

$$\begin{aligned} -6x + 1 &\le -5x - 2 \\ -x + 1 &\le -2 \\ -x &\le -3 \\ x &\ge 3 \end{aligned}$$

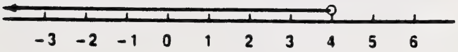
Solution set is $\{x | x \ge 3\}$.



(b)

$$\begin{aligned} 3x + 1 &> 5x - 7 \\ -2x + 1 &> -7 \\ -2x &> -8 \\ x &< 4 \end{aligned}$$

Solution set is $\{x | x < 4\}$.



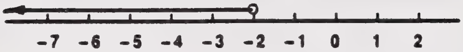
(c) $6x < 2(2x - 5) + 6$

$6x < 4x - 10 + 6$

$2x < -4$

$x < -2$

Solution set is $\{x|x < -2\}$.



(d) $3x - 3(2x - 5) < 2x - 5$

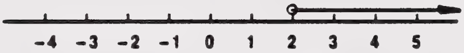
$3x - 6x + 15 < 2x + 5$

$-3x + 15 < 2x + 5$

$-5x < -10$

$x > 2$

Solution set is $\{x|x > 2\}$.



(e) $\frac{x}{2} + 2 < \frac{x}{4} + 1$

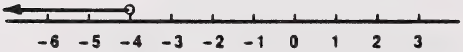
$4\left(\frac{x}{4}\right) + 2(4) < \frac{x}{4}(4) + 1(4)$

$2x + 8 < x + 4$

$x + 8 < 4$

$x < -4$

Solution set is $\{x|x < -4\}$.



(f)

$$\frac{x+1}{2} + 2 \geq \frac{5x-3}{3}$$

$$6\left(\frac{x+1}{2}\right) + 2(6) \geq \left(\frac{5x-3}{3}\right)6$$

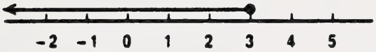
$$3x + 3 + 12 \geq 10x - 6$$

$$3x + 15 \geq 10x - 6$$

$$-7x \geq -21$$

$$x \leq 3$$

Solution set is $\{x \mid x \leq 3\}$.



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$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a = 1,\ b = -5,\ c = 3$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{25 - 12}}{2}$$

$$x = \frac{5 \pm \sqrt{13}}{2}$$

$$x = \frac{5 \pm 3.606}{2}$$

$$x = \frac{8.606}{2} \text{ or } \frac{1.394}{2}$$

$x = 4.303$ or 0.697 The solution set is $\{4.303, 0.697\}$.

Pages 48-52

Exercise – Quadratic Equations in One Variable

1. (a) o (b) $7; -7$ (c) $\{0, -5\}$

(d) $ax^2 + bx + c = 0$ (e) discriminant (f) equivalent, $\{-8, 8\}$

(g) zero (h) equal (i) absolute values, sign

2. (a)

$$9x^2 - 16 = 0$$
$$9x^2 = 16$$
$$x^2 = \frac{16}{9}$$
$$x = \pm\sqrt{\frac{16}{9}}$$
$$x = \pm\frac{4}{3}$$

Solution set is $\left\{\frac{-4}{3}, \frac{4}{3}\right\}$.

(b)

$$x^2 + 8x = -15$$
$$x^2 + 8x + 15 = 0$$
$$(x + 3)(x + 5) = 0$$
$$x + 3 = 0 \quad \text{or} \quad x + 5 = 0$$
$$x = -3 \qquad\qquad x = -5$$

Solution set is $\{-3, -5\}$.

(c)

$$x^2 + 4x = 3$$
$$x^2 + 4x - 3 = 0$$
$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)}$$
$$x = \frac{-4 \pm \sqrt{16 + 12}}{2}$$
$$x = \frac{-4 \pm \sqrt{28}}{2}$$
$$x = \frac{-4 \pm 5.292}{2}$$
$$x = -4.646 \quad \text{or} \quad 0.646$$

Solution set is $\{-4.646, 0.646\}$.

(d)

$$6m^2 + m = 0$$
$$m(6m + 1) = 0$$
$$m = 0 \quad \text{or} \quad 6m + 1 = 0$$
$$m = 0 \quad \text{or} \qquad m = -\frac{1}{6}$$

Solution set is $\left\{0, -\frac{1}{6}\right\}$.

(e)

$$2x^2 + 9 = 0$$
$$2x^2 = -9$$
$$x^2 = \frac{-9}{2}$$
$$x = \pm\sqrt{\frac{-9}{2}}$$

no real roots

Solution set is ϕ.

(f)

$$2x^2 = 3 - 5x$$
$$2x^2 + 5x - 3 = 0$$
$$(2x - 1)(x + 3) = 0$$
$$2x - 1 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = \frac{1}{2} \quad \text{or} \qquad x = -3$$

Solution set is $\left\{\frac{1}{2}, -3\right\}$.

(g) $x^2 + 25 = 10x$

$x^2 - 10 + 25 = 0$

$(x - 5)(x - 5) = 0$

$x - 5 = 0$

$x = 5$

Solution set is $\{5\}$.

(h) $x^2 + x - 11 = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-11)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{1 + 44}}{2}$$

$$x = \frac{-1 \pm \sqrt{45}}{2}$$

$$x = \frac{-1 \pm 6.708}{2}$$

$x = -3.854$ or 2.854

Solution set is $\{-3.854, 2.854\}$.

(i) $x^2 + x + 2 = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{-7}}{2}$$

no real roots

Solution set is ϕ.

(j) $4m = m^2$

$m^2 - 4m = 0$

$m(m - 4) = 0$

$m = 0$ or $m - 4 = 0$

$m = 0$ or $m = 4$

Solution set is $\{0, 4\}$.

3. (a) $x^2 = -2x + 3$

$x^2 + 2x - 3 = 0$

$b^2 - 4ac = 2^2 - 4(1)(-3)$

$= 4 + 12$

$= 16$

Roots are real and distinct.

(b) $-x^2 + x = 1$

$x^2 - x + 1 = 0$

$b^2 - 4ac = (-1)^2 - 4(1)(1)$

$= 1 - 4$

$= -3$

Roots are not real.

(c) $x^2 - 3\sqrt{2}x + 5 = 0$

$$\begin{aligned} b^2 - 4ac &= (-3\sqrt{2})^2 - 4(1)(5) \\ &= 18 - 20 \\ &= -2 \end{aligned}$$

Roots are not real.

(d) $4x^2 + 4x = -1$

$4x^2 + 4x + 1 = 0$

$$\begin{aligned} b^2 - 4ac &= 4^2 - 4(4)(1) \\ &= 16 - 16 \\ &= 0 \end{aligned}$$

Roots are real and equal.

4. Then $b^2 - 4ac < 0$

$$\begin{aligned} (-8)^2 - 4(m)(-9) &< 0 \\ 64 + 36m &< 0 \\ 36m &< -64 \\ m &< \frac{-64}{36} \\ m &< \frac{-16}{9} \end{aligned}$$

Roots are not real when $m < \dfrac{-16}{9}$.

5. (a) false (b) false (c) true (d) true

(e) false (f) true (g) true (h) false

(i) false (j) true (k) false

6.

$$\begin{aligned} (x - 7)(x - 3) &= 0 \\ x^2 - 7x - 3x + 21 &= 0 \\ x^2 - 10x + 21 &= 0 \end{aligned}$$

7. $2x^2 + 4x + (2 - k - k^2) = 0$

$a = 2,\ b = 4,\ c = 2 - k - k^2$

$$\begin{aligned} b^2 - 4ac &= 0 \\ 4^2 - 4(2)(2 - k - k^2) &= 0 \\ 16 - 16 + 8k + 8k^2 &= 0 \\ 8k^2 + 8k &= 0 \\ k^2 + k &= 0 \\ k(k + 1) &= 0 \end{aligned}$$

$k = 0$ or $k = -1$

8.

$$\begin{aligned} x(13 - x) &= 40 \\ 13x - x^2 &= 40 \\ x^2 - 13x + 40 &= 0 \\ (x - 5)(x - 8) &= 0 \end{aligned}$$

$x = 5$ or 8

The length is 8 units and the width is 5 units.

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$$\frac{1}{x} + \frac{x - 1}{x(x + 2)} = \frac{-x}{x + 2} \qquad x \neq 0, \quad x \neq -2$$

Multiply every term by x(x + 2).

$$x(x + 2)\left(\frac{1}{x}\right) + x(x + 2)\left[\frac{x - 1}{x(x + 2)}\right] = x(x + 2)\left(\frac{-x}{x + 2}\right)$$

Cancel like factors where possible.

$$\frac{x(x + 2)}{x} + \frac{x(x + 1)(x - 1)}{x(x + 2)} = \frac{x(x + 2)(-x)}{(x + 2)}$$

$$(x + 2) + (x - 1) = x(-x)$$
$$2x + 1 = -x^2$$
$$x^2 + 2x + 1 = 0$$
$$(x + 1)(x + 1) = 0$$
$$x = -1$$

Is this value among the permissible values of x? **Yes**

Solution set is $\{-1\}$.

$$\frac{3x^2}{3x + 1} - 2 = \frac{2x + 1}{3x + 1}, \quad x \neq -\frac{1}{3}$$

Multiply every term by (3x + 1).

$$(3x + 1)\left(\frac{3x^2}{3x + 1}\right) - 2(3x + 1) = \frac{(3x + 1)(2x + 1)}{(3x + 1)}$$

Cancel like factors where possible.

$$\frac{\cancel{(3x + 1)}(3x^2)}{\cancel{(3x + 1)}} - 2(3x + 1) = \frac{\cancel{(3x + 1)}(2x + 1)}{\cancel{(3x + 1)}}$$

$$3x^2 - 6x - 2 = 2x + 1$$
$$3x^2 - 8x - 3 = 0$$
$$(3x + 1)(x - 3) = 0$$

If $3x + 1 = 0$

$x = -\frac{1}{3}$

If $x - 3 = 0$

$x = 3$

Which of these roots is not among the permissible values of x? $x = -\frac{1}{3}$

Solution set is $\{3\}$.

page 57

$x = 2 - \sqrt{2x - 5}$

Isolate the radical term on the right side.

$x - 2 = -\sqrt{2x - 5}$

Square both sides.

$$\begin{aligned} (x - 2)^2 &= (-\sqrt{2x - 5}\,)^2 \\ x^2 - 4x + 4 &= 2x - 5 \\ x^2 - 6x + 9 &= 0 \\ (x - 3)(x - 3) &= 0 \\ x &= 3 \end{aligned}$$

$x = 3$ is a possible root of the original equation.

Check

If $X = 3$.

x	$2 - \sqrt{2x - 5}$
3	$2 - \sqrt{2(3) - 5}$
	$2 - \sqrt{6 - 5}$
	$2 - \sqrt{1}$
	$2 - 1$
	1

Is this value of x a root? No

The solution set is ϕ.

Page 59

$$\begin{aligned} \sqrt{2x + 3} &= \sqrt{x + 1} + 1 \\ (\sqrt{2x + 3}\,)^2 &= (\sqrt{x + 1} + 1)^2 \\ 2x + 3 &= (\sqrt{x + 1}\,)^2 + 2(\sqrt{x + 1}\,)(1) + 1^2 \\ 2x + 3 &= x + 1 + 2\sqrt{x + 1} + 1 \\ 2x + 3 &= x + 2 + 2\sqrt{x + 1} \\ x + 1 &= 2\sqrt{x + 1} \\ (x + 1)^2 &= (2\sqrt{x + 1}\,)^2 \\ x^2 + 2x + 1 &= 4(x + 1) \\ x^2 + 2x + 1 &= 4x + 4 \\ x^2 - 2x - 3 &= 0 \\ (x - 3)(x + 1) &= 0 \end{aligned}$$

If $(x - 3) = 0$ If $(x + 1) = 0$

$x = 3$ $x = -1$

Check: If $x = 3$.

$\sqrt{2x + 3} - \sqrt{x + 1}$	1
$\sqrt{2(3) + 3} - \sqrt{3 + 1}$	
$\sqrt{6 + 3} - \sqrt{4}$	
$\sqrt{9} - \sqrt{4}$	
$3 - 2$	
1	

Is this value of x a root? Yes

If $x = -1$.

$\sqrt{2x + 3} - \sqrt{x + 1}$	1
$\sqrt{2(-1) + 3} - \sqrt{-1 + 1}$	
$\sqrt{-2 + 3} - \sqrt{0}$	
$\sqrt{1} - \sqrt{0}$	
$1 - 0$	
1	

Is this value of x a root? Yes

Conclusion: The solution set of the equation $\sqrt{2x + 3} - \sqrt{x + 1} = 1$ is $\{3, -1\}$.

Pages 60-62 **Exercise – Fractional and Radical Equations**

1. $\frac{x+3}{x} - x = 5 + \frac{3}{x}, \quad x \neq 0$

$$\frac{x+3}{x}(x) - x(x) = 5(x) + \frac{3}{x}(x)$$

$$x + 3 - x^2 = 5x + 3$$
$$x^2 + 4x = 0$$
$$x(x + 4) = 0$$

$x = 0$ or $x + 4 = 0$

$x = 0$ or $x = -4$

Which of the roots is not among the permissible values of x? **0**

Solution set is $\{-4\}$.

2. $(\sqrt{2x-3})^2 = 5^2$

$$2x - 3 = 25$$
$$2x = 28$$
$$x = 14$$

Check: If $x = 14$

$\sqrt{2x-3} + 4$	9
$\sqrt{2(14)-3} + 4$	9
$\sqrt{28-3} + 4$	
$\sqrt{25} + 4$	
$5 + 4$	
9	

Solution set is $\{14\}$.

3. $1 + \frac{12}{x^2-4} = \frac{3}{x-2}, \quad x \neq 2, \quad x \neq -2$

$$1(x+2)(x-2) + \left[\frac{12}{(x-2)(x+2)}\right](x-2)(x+2) = \left(\frac{3}{x-2}\right)(x-2)(x+2)$$

$$x^2 - 4 + 12 = 3x + 6$$
$$x^2 - 3x + 2 = 0$$
$$(x-2)(x-1) = 0$$
$$x = 2 \text{ or } 1$$

Which root is not among the permissible values of x? $x = 2$

Solution set is $\{1\}$.

4. $\frac{3}{x-3} + \frac{1}{x-1} = \frac{x}{x^2-4x+3}$, $x \neq 3$, and $x \neq 1$

$$\left(\frac{3}{x-3}\right)(x-3)(x-1) + \left(\frac{1}{x-1}\right)(x-3)(x-1) = \frac{x(x-3)(x-1)}{(x-3)(x-1)}$$

$$3(x-1) + (x-3) = x$$

$$3x - 3 + x - 3 = x$$

$$3x = 6$$

$$x = 2$$

Solution set is {2}.

5. $(-2\sqrt{y+1})^2 = (y-2)^2$

$$4(y+1) = y^2 - 4y + 4$$

$$4y + 4 = y^2 - 4y + 4$$

$$y^2 - 8y = 0$$

$$y(y-8) = 0$$

$y = 0$ or $y - 8 = 0$

$y = 0$ or 8

Solution set is {0}.

If y = 0,	$2 - y - 2\sqrt{y+1}$	0
	$2 - 0 - 2\sqrt{0+1}$	
	$2 - 2\sqrt{1}$	
	$2 - 2 \times 1$	
	$2 - 2$	
	0	
If y = 8,	$2 - 8 - 2\sqrt{8+1}$	0
	$2 - 8 - 2\sqrt{9}$	
	$-6 - 2(3)$	
	$-6 - 6$	
	-12	

6. $(\sqrt{x+2})^2 = (1 + \sqrt{3-x})^2$

$$x + 2 = 1 + 2\sqrt{3-x} + 3 - x$$

$$(2x-2)^2 = (2\sqrt{3-x})^2$$

$$4x^2 - 8x + 4 = 4(3-x)$$

$$4x^2 - 8x + 4 = 12 - 4x$$

$$4x^2 - 4x - 8 = 0$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \text{ or } -1$$

Solution set is {2}.

If x = 2,	$\sqrt{x+2}$	$1 + \sqrt{3-x}$
	$\sqrt{2+2}$	$1 + \sqrt{3-2}$
	$\sqrt{4}$	$1 + \sqrt{1}$
	2	$1 + 1$
		2
If x = −1,	$\sqrt{-1+2}$	$1 + \sqrt{3-(-1)}$
	$\sqrt{1}$	$1 + \sqrt{4}$
	1	$1 + 2$
		3

END OF LESSON F

LESSON G

Page 2

$x = 2$ and $y = 4$ (since $2 + 4 = 6$)

$x = -1$ and $y = 7$ (since $-1 + 7 = 6$)

$x = 0$ and $y = 6$ (since $0 + 6 = 6$)

(−3,9), (3,3), (2, 4), (−1,7), (0,6)

(12,6), (7,1), (15,9), (10,4), (6,0)

Pages 9-10

1. (2,7), (3,1), (5,2), (6,3) (accept other answers)
2. (−5,6), (−3,1), (−5,2), (−6,3) (accept other answers)
3. (−3,−5), (−3,−1), (−5,−2), (−6,−3) (accept other answers)
4. (6,−3), (3,−1), (5,−2), (6,−3) (accept other answers)

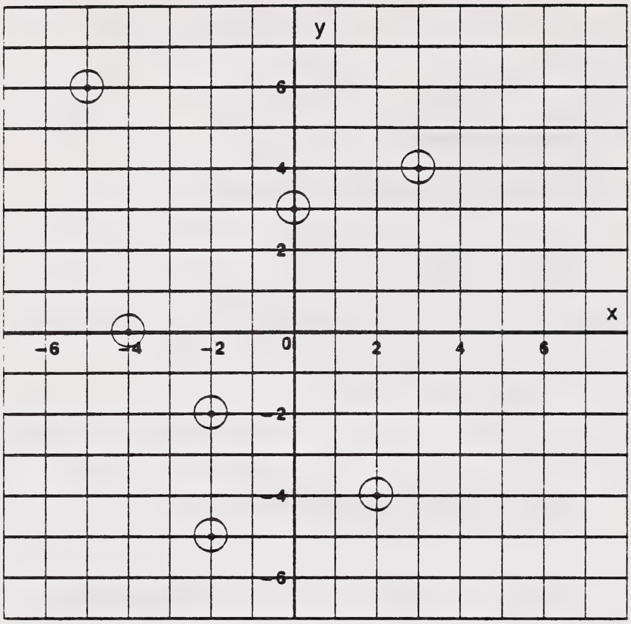
1. (0,7), (0,−1), (0,3), (0,−4) (accept other answers)
2. (6,0), (3,0), (−2,0), (−4,0) (accept other answers)

Pages 17-23

1. (a) ordered pairs, straight line
   (b) origin
   (c) fourth
   (d) coordinates, x-coordinate or abscissa, y-coordinate or ordinate
   (e) y
   (f) negative, positive
   (g) 2
   (h) points, coordinate

2. (b) (2,2), (6,−2)
   (c) (1,2), (3,6), (−1,−2)
   (d) (−1,−2), (−2,0)
   (e) (1,2), (−1,−2)
   (f) (−1,−2)

3.



4. (a) $(-8,6)$ (b) $(-4,2)$ (c) $(0,5)$
(d) $(6,9)$ (e) $(7,0)$ (f) $(5,-6)$
(g) $(0,-9)$ (h) $(-3,-6)$ (i) $(-7,-3)$
(j) $(-9,0)$

5. (a) Quadrant III (b) Quadrant I (c) x-axis
(d) Quadrant II (e) Quadrant IV (f) y-axis
(g) x-axis (h) Quadrant I (i) Quadrant III

6. (a) $y = 2x + 1$

(b) When $x = -1$, $y = 2(-1) + 1 = -2 + 1 = -1$

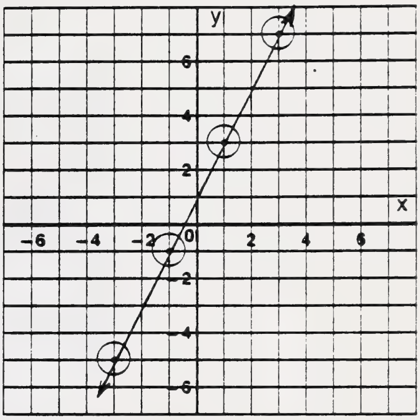
When $x = 1$, $y = 2(1) + 1 = 2 + 1 = 3$

When $x = 3$, $y = 2(3) + 1 = 6 + 1 = 7$

6. (c)

x	−3	−1	1	3
y	−5	−1	3	7

Yes.



7. (a) $2y = x + 3$

$$y = \frac{x + 3}{2}$$

(b) (ii) $y = \frac{-5 + 3}{2} = -1$

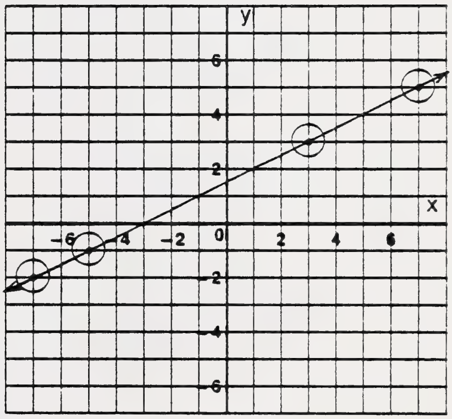
(iii) $y = \frac{3 + 3}{2} = 3$

(iv) $y = \frac{7 + 3}{2} = 5$

(c)

x	−7	−5	3	7
y	−2	−1	3	5

Yes.



8. (a) $y = -2x + 3$

(b) $x - y = 3$

$-y = -x + 3$

$y = x - 3$

(c) $2y = -x + 4$

$$y = \frac{-x + 4}{2}$$

(d) $y = -x + 3$

(e) $4y = -5x - 2$

$$y = \frac{-5x - 2}{4}$$

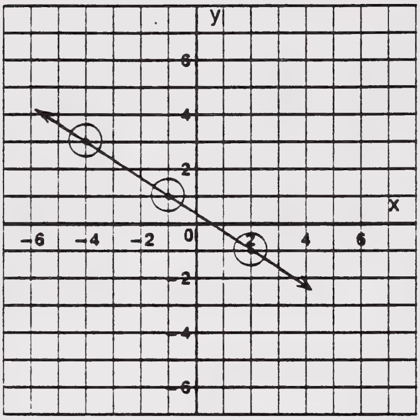
(f) $2y = 4x + 6$

$$y = \frac{4x + 6}{2}$$

$y = 2x + 3$

9. (a) $2x + 3y = 1$

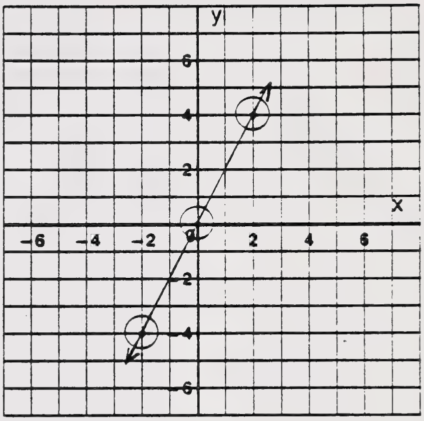
$$y = \frac{-2x + 1}{3}$$



x	-4	-1	2
y	3	1	-1

(b) $2x - y = 0$

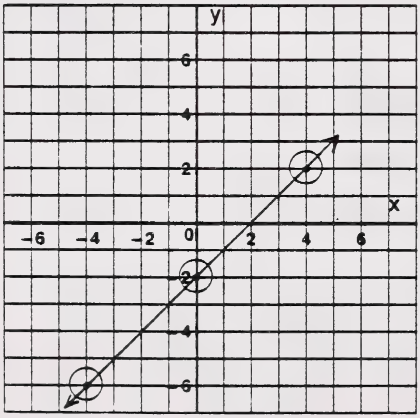
$$y = 2x$$



x	-2	0	2
y	-4	0	4

(c) $x - y = 2$

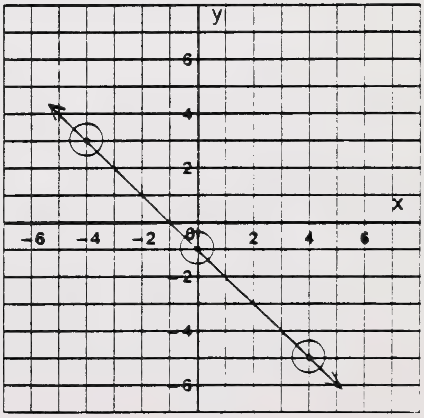
$$y = x - 2$$



x	-4	0	4
y	-6	-2	2

(d) $x + y = -1$

$$y = -x - 1$$



x	-4	0	4
y	3	-1	-5

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2. Domain: $\{2\}$, Range: $\{-2,-1,0,1,2\}$

3. Domain: $\{-3,-2,-1,1,2,3\}$, Range: $\{1,4,9\}$

Pages 39-44

Exercise – Relations and Functions

1. (a) domain, range (b) relation, one
   (c) function (d) y, x
   (e) $-2(-1)^2 + (-1) = -2 - 1 = -3$ (f) $\{1,2,3\}$, $\{2,3\}$
   (g) -3 and 3 (h) 3 (i) 0, y

2. (a) (ii) Range: $\{y \mid -2 \leq y \leq 2\}$
   (iii) x-intercepts: -2 and 2
   (iv) y-intercepts: -2 and 2

   (b) (i) Domain: $\{x \mid x \leq -2 \text{ or } x \geq 2\}$
   (ii) Range: R
   (iii) x-intercepts: -2 and 2
   (iv) y-intercepts: none

   (c) (i) Domain: $\{x \mid x \leq 0\}$
   (ii) Range: R
   (iii) x-intercepts: 0
   (iv) y-intercepts: 0

   (d) (i) Domain: R
   (ii) Range: $\{2\}$
   (iii) x-intercepts: none
   (iv) y-intercepts: 2

   (e) (i) Domain: $\{x \mid 0 \leq x \leq 6\}$
   (ii) Range: $\{y \mid -5 \leq y \leq 1\}$
   (iii) x-intercepts: 0.8 and 5.2
   (iv) y-intercepts: -2

   (f) (i) Domain: R
   (ii) Range: $\{y \mid y \geq -4\}$
   (iii) x-intercepts: -4 and 0
   (iv) y-intercepts: 0

3. (d) and (f)

4. (a) $\{(x,y) \mid x^2 + y^2 = 4\}$

Domain	**Range**
Make y the subject.	Make x the subject
$y^2 = 4 - x^2$	$x^2 = 4 - y^2$
$\lvert y \rvert = \sqrt{4 - x^2}$	$\lvert x \rvert = \sqrt{4 - y^2}$
$y \in R$ if $4 - x^2 \geq 0$	$x \in R$ when $4 - y^2 \geq 0$
$-x^2 \geq -4$	$-y^2 \geq -4$
$x^2 \leq 4$	$y^2 \leq 4$
$\lvert x \rvert \leq 2$	$\lvert y \rvert \leq 2$
Domain is $\{x \mid -2 \leq x \leq 2\}$	Range is $\{y \mid -2 \leq y \leq 2\}$

5. (a) $\{(x,y) \mid y = x^2 + 7x\}$

x-intercepts	**y-intercepts**
Let $y = 0$.	Let $x = 0$.
$x^2 + 7x = 0$	$y = (0)^2 + 7(0)$
$x(x + 7) = 0$	$y = 0 + 0$
$x = 0$ or $x + 7 = 0$	$y = 0$
$x = 0$ or -7	
x-intercepts are 0 and -7	y-intercept is 0.

(b) $\{(x,y) \mid 9x^2 + 6y^2 = 36\}$

x-intercepts	**y-intercepts**
Let $y = 0$.	Let $x = 0$.
$9x^2 + 6(0)^2 = 36$	$9(0)^2 + 6y^2 = 36$
$9x^2 = 36$	$6y^2 = 36$
$x^2 = 4$	$y^2 = 6$
$x = \pm 2$	$y = \pm\sqrt{6}$
x-intercepts are -2 and 2.	y-intercepts are $-\sqrt{6}$ and $\sqrt{6}$.

6. (i) Domain

$y^2 = x^2 + 4$

$y = \pm\sqrt{x^2 + 4}$

$y \in R$ when $x^2 + 4 \geq 0$.

This is always true.

Domain is R.

(ii) Range

$y^2 - x^2 = 4$

$x^2 = y^2 - 4$

$|x| = \sqrt{y^2 - 4}$

$x \in R$ when $y^2 - 4 \geq 0$

$y^2 \geq 4$

$|y| \geq 2$

Range is $\{y | y \leq -2 \text{ or } y \geq 2\}$.

(iii) x-intercepts

Let $y = 0$

$0^2 - x^2 = 4$

$x^2 = -4$

$x = \pm\sqrt{-4}$

= no real number

No x-intercepts

(iv) y-intercepts

Let $x = 0$

$y^2 - 0^2 = 4$

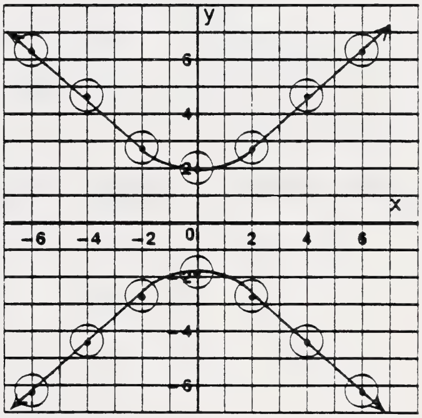
$y^2 = 4$

$y = \pm 2$

y-intercepts are −2 and 2.

(v) Graph

x	−6	−4	−2	0	2	4	6
y	±6.3	±4.5	±2.8	±2	±2.8	±4.5	±6.3



7. (a) 0

(b) $2(-1)^2 - (-1) = 2 + 1 = 3$

(c) $2a^2 - a$

(d) $2(a + 1)^2 - (a + 1) = 2a^2 + 4a + 2 - a - 1 = 2a^2 + 3a + 1$

(e) $2y^2 - y$

(f) $2(2)^2 - 2 = 8 - 2 = 6$

8. (a) 1 (b) 16

(c) $(\sqrt{3})^2 + 2 = 3 + 2 = 5$

(d) $4 - (a + 1) = 4 - a - 1 = 3 - a$

(e) $(4c)^2 + 2 = 16c^2 + 2$

(f) $\dfrac{(4 - d) - (4 - c)}{d - c} = \dfrac{4 - d - 4 + c}{d - c} = \dfrac{-d + c}{d - c} = \dfrac{-1(d - c)}{(d - c)} = -1$

(g) $f(4 - 2) = f(2) = 2^2 + 2 = 4 + 2 = 6$

(h) $g(x^2 + 2) = 4 - (x^2 + 2) = 4 - x^2 - 2 = 2 - x^2$

Pages 56-59 Exercise – Linear Functions

1. (a) x, 0, constant (b) parallel

(c) $-2, -2, -1$, falls (d) $\{-4\}$, R

(e) R, R (f) $y = -\frac{1}{2}x + 7$

(g) $(0,1)$ (h) slope

2. (a) $\frac{4}{5}$ (b) $\dfrac{4 - 4}{-3 - 2} = \dfrac{0}{-5} = 0$

(c) $= \dfrac{-3 - 5}{-2 - (-1)} = \dfrac{-8}{-1} = 8$ (d) $\dfrac{4 - (-1)}{-2 - 3} = \dfrac{5}{-5} = -1$

(e) $\dfrac{+3 - (-6)}{2 - 2} = \dfrac{9}{0} =$ undefined (f) $\dfrac{-4 - 6}{-2 - 5} = \dfrac{-10}{-7} = \dfrac{10}{7}$

3. (a)

$$y = 4x + 1$$
$$\text{slope} = 4$$
$$y\text{-intercept} = 1$$

(b)

$$y = 2x - 3$$
$$\text{slope} = 2$$
$$y\text{-intercept} = -3$$

(c)

$$3y = x - 1$$
$$y = \frac{x}{3} - \frac{1}{3}$$
$$\text{slope} = \frac{1}{3}$$
$$y\text{-intercept} = -\frac{1}{3}$$

(d)

$$-3y = 2x + 5$$
$$y = \frac{-2}{3}x - \frac{5}{3}$$
$$\text{slope} = -\frac{2}{3}$$
$$y\text{-intercept} = -\frac{5}{3}$$

4. (a) $y = -x + 5$

(b)

$$\text{slope} = \frac{-2 - (-4)}{-4 - 3}$$
$$= \frac{+2}{-7} = \frac{-2}{7}$$
$$y - y_1 = m(x - x_1)$$
$$y - (-2) = \frac{-2}{7}[x - (-4)]$$
$$y + 2 = \frac{-2}{7}(x + 4)$$
$$7y + 14 = -2x - 8$$
$$7y = -2x - 22$$
$$y = \frac{-2}{7}x - \frac{22}{7}$$

(c)

$$y - y_1 = m(x - x_1)$$
$$y - (-4) = \frac{5}{3}[x - (-3)]$$
$$y + 4 = \frac{5}{3}(x + 3)$$
$$3y + 12 = 5x + 15$$
$$3y = 5x + 3$$
$$y = \frac{5}{3}x + 1$$

(d)

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = -1(x - 1)$$
$$y + 3 = -x + 1$$
$$y = -x - 2$$

(e)

$$y = mx + b$$
$$y = 4x + 2$$

(f)

$$\frac{y - y_1}{x - x_1} = \frac{3 - 7}{7 - 3} = \frac{-4}{4} = -1$$
$$y - y_1 = m(x - x_1)$$
$$y - 3 = -1(x - 7)$$
$$y - 3 = -x + 7$$
$$y = -x + 10$$

END OF LESSON G

LESSON H

Page 14

$y = 9x^2 - 6x + 4$	$y = (9x^2 - 6x) + 4$	$y = 9\left(x^2 - \frac{2}{3}x\right) + 4$
$y = -3x^2 + 6x - 1$	$y = (-3x^2 + 6x) - 1$	$y = -3(x^2 - 2x) - 1$
$y = 6x^2 + 7x - 3$	$y = (6x^2 + 7x) - 3$	$y = 6\left(x^2 + \frac{7}{6}x\right) - 3$
$y = -x^2 - 2x + 9$	$y = (-x^2 - 2x) + 9$	$y = -1(x^2 + 2x) + 9$
$y = -5x^2 - 15x + 9$	$y = (-5x^2 - 15x) + 9$	$y = -5(x^2 + 3x) + 9$
$y = 3x^2 - 9x - 4$	$y = (3x^2 - 9x) - 4$	$y = 3(x^2 - 3x) - 4$

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$y = (x^2 + 8x) + 3$	$\left(\frac{8}{2}\right)^2 = 16$	$a = 1$	$y + 1(16) = (x^2 + 8x + 16) + 3$
$y = 3(x^2 + 12x) - 1$	$\left(\frac{12}{2}\right)^2 = 6^2 = 36$	$a = 3$	$y + 3(36) = 3(x^2 + 12x + 36) - 1$
$y = -2(x^2 - 5x) + \frac{3}{4}$	$\left(\frac{-5}{2}\right)^2 = \frac{25}{4}$	$a = -2$	$y + (-2)\left(\frac{25}{4}\right) = -2\left(x^2 - 5x + \frac{25}{4}\right) + \frac{3}{4}$

Page 16

$y + (-4)\left(\frac{9}{4}\right) = -4\left(x^2 - 3x + \frac{9}{4}\right) + 8$ $y - 9 = -4\left(x - \frac{3}{2}\right)^2 + 8$ $y = -4\left(x - \frac{3}{2}\right)^2 + 8 + 9$ $y = -4\left(x - \frac{3}{2}\right)^2 + 17$	$x = \frac{3}{2}$	$\left(\frac{3}{2}, 17\right)$
$y + 9\left(\frac{1}{9}\right) = 9\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + 4$ $y + 1 = 9\left(x - \frac{1}{3}\right)^2 + 4$ $y = 9\left(x - \frac{1}{3}\right)^2 + 4 - 1$ $y = 9\left(x - \frac{1}{3}\right)^2 + 3$	$x = \frac{1}{3}$	$\left(\frac{1}{3}, 3\right)$

Pages 26-31 **Exercise – Quadratic Function**

1. (a) relation, function (b) minimum

(c) 0 , 0 , origin (d) $(-3, -1)$

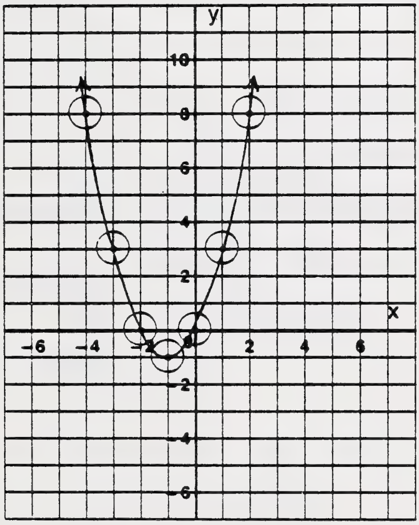
(e) $\{y | y \leq -2\}$ (f) vertex

(g) $x = -1$ (h) 1 and -1 , -1

(i) x (j) downwards

2.

x	−4	−3	−2	−1	0	1	2
y	8	3	0	−1	0	3	8



(a) The graph opens **upward.**

(b) The y-intercept is **0.**

(c) The x-intercepts are **−2** and **0.**

(d) The equation of the axis of symmetry is $\mathbf{x = -1}$.

(e) The coordinates off the vertex are $\mathbf{(-1, -1)}$.

(f) The range of the function is $\{y | y \geq -1\}$.

(g) The domain of the function is $\{x | x \in \mathbf{R}\}$.

3. (a) true (b) false (c) true (d) false

(e) true (f) false (g) false (h) true

(i) true (j) false (k) true (l) true

(m) false

4. (a) q_1, q_2, q_3, q_6, q_7

(b) q_2, q_3

(c) q_1, q_5

(d) q_2, q_3

(e) all of them

(f) q_3, q_4, q_8

(g) q_7

(h) q_1, q_2, q_5, q_7

(i) q_1, q_4, q_5, q_7

(j) q_6

5. (a) $y = -2(x - 0)^2 + 7$

Axis of symmetry is $x = 0$.

Coordinates of vertex are $(0,7)$.

Range is $\{y | y \leq 7\}$

(b) $y = 7(x - 0)^2 + 0$

Axis of symmetry is $x = 0$.

Coordinates of vertex are $(0,0)$.

Range is $\{y | y \geq 0\}$

(c)
$$y + 3(1) = 3(x^2 + 2x + 1)$$
$$y + 3 = 3(x + 1)^2$$
$$y = 3(x + 1)^2 - 3$$

Axis of symmetry is $x = -1$.

Coordinates of vertex are $(-1, -3)$.

Range is $\{y | y \geq -3\}$

(d)
$$y + \frac{1}{4} = \left(x^2 + x + \frac{1}{4}\right) - 7$$
$$y + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2 - 7$$
$$y = \left(x + \frac{1}{2}\right)^2 - 7\frac{1}{4}$$

Axis of symmetry is $x = -\frac{1}{2}$.

Coordinates of vertex are $\left(-\frac{1}{2}, -7\frac{1}{4}\right)$.

Range is $\left\{y | y \geq -7\frac{1}{4}\right\}$

(e)
$$y = 2(x^2 - 4x) + 3$$
$$y + 2(4) = 2(x^2 - 4x + 4) + 3$$
$$y + 8 = 2(x - 2)^2 + 3$$
$$y = 2(x - 2)^2 - 5$$

Axis of symmetry is $x = 2$.

Coordinates of vertex are $(2, -5)$.

Range is $\{y | y \geq -5\}$

(f)

$$y = -2\left(x^2 - \frac{3}{2}x\right) + 1$$

$$y + (-2)\left(\frac{9}{16}\right) = -2\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) + 1$$

$$y = -2\left(x - \frac{3}{4}\right)^2 + \frac{17}{8}$$

Axis of symmetry is $x = \frac{3}{4}$.

Coordinates of vertex are $\left(\frac{3}{4}, \frac{17}{8}\right)$.

Range is $\left\{y \middle| y \leq \frac{17}{8}\right\}$.

6. (a) $\{(x,y)|y = 2x^2 - 4\}$

$a = 2,\ b = 0,\ c = -4$

$$\frac{-b}{2a} = \frac{0}{2(2)} = \frac{0}{4} = 0$$

$$\frac{4ac - b^2}{4a} = \frac{4(2)(-4) - 0}{4(2)} = -4$$

Axis of symmetry is $x = 0$.

Coordinates of vertex are $(0, -4)$.

Range is $\{y|y \geq -4\}$.

(b) $\{(x,y)|y = -x^2 - x\}$

$a = -1,\ b = -1,\ c = 0$

$$\frac{-b}{2a} = \frac{-(-1)}{2(-1)} = \frac{1}{-2} = -\frac{1}{2}$$

$$\frac{4ac - b^2}{4a} = \frac{4(-1)(0) - (-1)^2}{4(-1)} = \frac{0 - 1}{-4} = \frac{1}{4}$$

Axis of symmetry is $x = -\frac{1}{2}$.

Coordinates of vertex are $\left(-\frac{1}{2}, \frac{1}{4}\right)$.

Range is $\left\{y \middle| y \leq \frac{1}{4}\right\}$.

(c) $\{(x,y)|y = -5x^2\}$

$a = -5, b = 0, c = 0$

$$\frac{-b}{2a} = \frac{-0}{2(-5)} = \frac{0}{-10} = 0$$

$$\frac{4ac - b^2}{4a} = \frac{4(-5)(0) - 0^2}{4(-5)} = \frac{0}{-20} = 0$$

Axis of symmetry is $x = 0$.

Coordinates of vertex are $(0,0)$.

Range is $\{y|y \leq 0\}$.

(d) $\{(x,y)|y = x^2 - 6x + 8\}$

$a = 1, b = -6, c = 8$

$$\frac{-b}{2a} = \frac{-(-6)}{2(1)} = \frac{6}{2} = 3$$

$$\frac{4ac - b^2}{4a} = \frac{4(1)(8) - (-6)^2}{4(1)} = \frac{32 - 36}{4} = -1$$

Axis of symmetry is $x = 3$.

Coordinates of vertex are $(3,-1)$.

Range is $\{y|y \geq -1\}$.

(e) $\{(x,y)|y = 2x^2 - 8x + 3\}$

$a = 2, b = -8, c = 3$

$$\frac{-b}{2a} = \frac{-(-8)}{2(2)} = \frac{8}{4} = 2$$

$$\frac{4ac - b^2}{4a} = \frac{4(2)(3) - (-8)^2}{4(2)} = \frac{-40}{8} = -5$$

Axis of symmetry is $x = 2$.

Coordinates of vertex are $(2,-5)$.

Range is $\{y|y \geq -5\}$.

(f) $\{(x,y) | y = -2x^2 + 3x + 1\}$

$a = -2,\ b = 3,\ c = 1$

$$\frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{-3}{-4} = \frac{3}{4}$$

$$\frac{4ac - b^2}{4a} = \frac{4(-2)(-1) - 3^2}{4(-2)} = \frac{-8 - 9}{-8} = \frac{17}{8}$$

Axis of symmetry is $x = \frac{3}{4}$.

Coordinates of vertex are $\left(\frac{3}{4}, \frac{17}{8}\right)$.

Range is $\left\{y | y \le \frac{17}{8}\right\}$.

7. (a) $\{(x,y) | y = x^2 - 4\}$

Let $x = 0$, $y = 0^2 - 4$

$y = -4$

y-intercept is -4.

Let $y = 0$, $0 = x^2 - 4$

$(x - 2)(x + 2) = 0$

$x = 2$ or -2

x-intercepts are 2 and -2.

The graph comes in contact with the x-axis twice.

(b) $\{(x,y) | y = x^2 - 2x + 1\}$

Let $x = 0$, $y = 0^2 - 2(0) + 1$

$y = 1$

y-intercept is 1.

Let $y = 0$, $0 = x^2 - 2x + 1$

$(x - 1)(x - 1) = 0$

$x = 1$

x-intercept is 1.

The graph comes in contact with the x-axis once.

(c) $\{(x,y) | y = x^2 + 3x - 1\}$

Let $x = 0$, $y = 0^2 + 3(0) - 1$

$y = -1$

y-intercept is -1.

Let $y = 0$, $0 = x^2 + 3x - 1$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-1)}}{2(1)}$$

$x = -3.3028$ or 0.3028

x-intercepts are -3.3028 and 0.3028.

The graph comes in contact with the x-axis twice.

(d) $\{(x,y) | y = x^2 - x + 5\}$

Let $x = 0$, $y = 0^2 - 0 + 5$

$y = 5$

y-intercept is 5.

Let $y = 0$, $0 = x^2 - x + 5$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{-19}}{2} = \text{not real}$$

no x-intercepts.

The graph never comes in contact with the x-axis

8. $a = -16$, $b = +120$, $c = 0$

$$t = \frac{-b}{2a} = \frac{-120}{2(-16)} = \frac{-120}{-32} = 3\frac{3}{4}$$

$$s = \frac{4ac - b^2}{4a} = \frac{4(-16)(0) - (120)^2}{4(-16)} = 225$$

The maximum height reached by the bullet is 225 feet and it takes $3\frac{3}{4}$ seconds to reach this height.

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upward , downward , $\dfrac{4ac - b^2}{4a}$, $\dfrac{4ac - b^2}{4a}$

When $x = -5$, $y = \ldots = 3$ or -1

When $x = 0$, $y = \ldots = 4$ or -2

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x	9	−8		−5		0	
y	1	2	0	3	−1	4	−2

3. $x = \dfrac{-32 - 4}{4} = \dfrac{-36}{4} = -9$

$y = \dfrac{2}{2} = 1$

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When $x = 0$, $y = \ldots = \pm 5$

When $x = \pm 1$, $y = \ldots \doteq \pm 4.9$

When $x = \pm 2$, $y = \ldots \doteq \pm 4.6$

When $x = \pm 3$, $y = \ldots = \pm\sqrt{16} = \pm 4$

When $x = \pm 4$, $y = \ldots = \pm\sqrt{25 - 16} = \sqrt{9} = \pm 3$

When $x = \pm 5$, $y = \ldots = \pm\sqrt{25 - 25} = \pm\sqrt{0} = 0$

x	0	±1	±2	±3	±4	±5
y	±5	±4.9	±4.6	±4	±3	0

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When $x = \pm 2$, $y = \ldots = \pm\dfrac{\sqrt{20}}{3} \doteq \pm\dfrac{4.473}{3} \doteq \pm 1.5$

When $x = \pm 2.5$, $y = \ldots = \dfrac{\sqrt{11}}{3} \doteq \pm\dfrac{3.317}{3} \doteq \pm 1.1$

When $x = \pm 3$, $y = \ldots = \pm\dfrac{\sqrt{36 - 36}}{3} = \pm\dfrac{\sqrt{0}}{3} = 0$

x	0	±1	±2	±2.5	±3
y	±2	±1.9	±1.5	±1.1	0

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$x = \pm 4$

x-intercepts are -4 and 4.

When $x = \pm 4$, $y = \ldots = 0$

When $x = \pm 5$, $y = \ldots = \pm\sqrt{9} = \pm 3$

When $x = \pm 8$, $y = \ldots = \pm\sqrt{48} = \pm 6.9$

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x	±4	±5	±6	±8
y	0	±3	±4.1	±6.9

$y^2 = 4$

$y = \pm 2$

The y-intercepts are -2 and 2.

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When $x = 0$, $y = \ldots = \dfrac{\pm 4}{2} = \pm 2$

When $x = \pm 1$, $y = \ldots \doteq \pm 2.1$

When $x = \pm 2$, $y = \ldots = \pm\dfrac{\sqrt{20}}{2} \doteq \pm\dfrac{4.472}{2} \doteq \pm 2.2$

When $x = \pm 4$, $y = \ldots = \pm\dfrac{\sqrt{16 + 16}}{2} = \pm\dfrac{\sqrt{32}}{2} \doteq \pm\dfrac{5.657}{2} \doteq \pm 2.8$

When $x = \pm 6$, $y = \ldots = \pm\dfrac{\sqrt{36 + 16}}{2} = \pm\dfrac{\sqrt{52}}{2} \doteq \pm\dfrac{7.211}{2} \doteq \pm 3.6$

x	0	±1	±2	±4	±6
y	±2	±2.1	±2.2	±2.8	±3.6

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When $x = -8$, $y = \frac{24}{-8} = -3$

When $x = -6$, $y = \frac{24}{-6} = -4$

When $x = -4$, $y = \frac{24}{-4} = -6$

When $x = -3$, $y = \frac{24}{-3} = -8$

When $x = 8$, $y = \frac{24}{8} = 3$

When $x = 6$, $y = \frac{24}{6} = 4$

When $x = 4$, $y = \frac{24}{4} = 6$

When $x = 3$, $y = \frac{24}{3} = 8$

x	−8	−6	−4	−3	3	4	6	8
y	−3	−4	−6	−8	8	6	4	3

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x	−8	−6	−4	−3	3	4	6	8
y	3	4	6	8	−8	−6	−4	−3

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Exercise – Some Quadratic Relations

1.

(b)	$3x^2 - 3y^2 = 1$	hyperbola
(c)	$x^2 + y^2 = \frac{1}{3}$	circle
(d)	$xy = 25$	hyperbola
(e)	$5x^2 + 7y^2 = 25$	ellipse
(f)	$x = y^2 - y - 6$	parabola
(g)	$y = 2x^2$	parabola
(h)	$xy = -16$	hyperbola
(i)	$x^2 + y^2 = 4$	circle
(j)	$y = 2x^2 + 4x$	parabola
(k)	$4x^2 - y^2 = 12$	hyperbola
(l)	$x = -3y^2 + 4$	parabola
(m)	$x^2 + y^2 = 5$	circle
(n)	$y = x^2 + 5$	parabola
(o)	$xy = 12$	hyperbola
(p)	$x^2 + 2y^2 = 4$	ellipse

2. (a) $\{(x,y) \mid x^2 + 4y^2 = 36\}$

(i) **ellipse**

(ii) x-intercepts

Let $y = 0$.

$$x^2 + 4(0)^2 = 36$$
$$x^2 = 36$$
$$x = \pm 6$$

x-intercepts are -6 and 6.

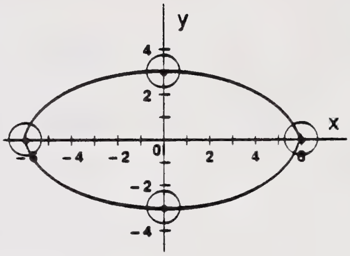
(iii) y-intercepts

Let $x = 0$.

$$0^2 + 4y^2 = 36$$
$$4y^2 = 36$$
$$y^2 = 9$$
$$y = \pm 3$$

y-intercepts are -3 and 3.

(iv) Graph



(v) The domain is $\{x \mid -6 \le x \le 6\}$ and the range is $\{y \mid -3 \le y \le 3\}$.

(b) $\{(x,y) \mid 2x^2 + 2y^2 = 32\}$

(i) **circle**

(ii) x-intercepts

Let $y = 0$.

$$2x^2 + 2(0)^2 = 32$$
$$2x^2 = 32$$
$$x^2 = 16$$
$$x = \pm 4$$

x-intercepts are -4 and 4.

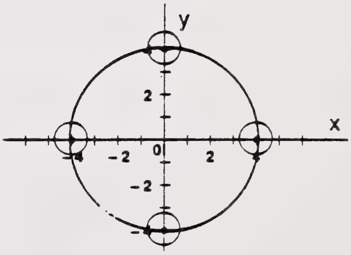
(iii) y-intercepts

Let $x = 0$.

$$2(0)^2 + 2y^2 = 32$$
$$2y^2 = 32$$
$$y^2 = 16$$
$$y = \pm 4$$

y-intercepts are -4 and 4.

(iv) Graph



(v) The domain is $\{x \mid -4 \le x \le 4\}$ and the range is $\{y \mid -4 \le y \le 4\}$.

(c) $\{(x,y)|xy = -16\}$

(i) **hyperbola**

(ii) x-intercepts

Let $y = 0$.

$x(0) = -16$

$0 \neq -16$

no real value

no x-intercept

(iii) y-intercepts

Let $x = 0$.

$0(y) = -16$

$0 \neq -16$

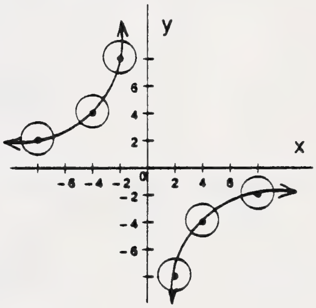
no real value

no y-intercept

$\{x|x \in R, x \neq 0\}$

(iv) Graph

x	-2	-4	-8	2	4	8
y	8	4	2	-8	-4	-2



$\{y|y \in R, y \neq 0\}$

(v) The domain is **all real numbers except zero** and the range is **all real numbers except zero**.

(d) $\{(x,y)|x = -y^2\}$

(i) **parabola**

(ii) x-intercepts

Let $y = 0$.

$x = -(0)^2$

$x = 0$

x-intercept is 0.

(iii) y-intercepts

Let $x = 0$.

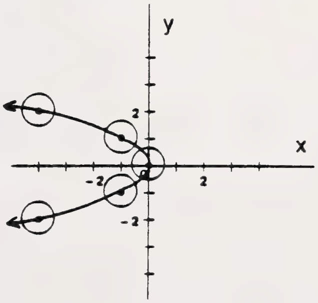
$0 = -y^2$

$y = 0$

y-intercept is 0.

(iv) Graph

x	0	-1	-1	-4	-4
y	0	-1	1	2	-2



(v) The domain is $\{x|x \leq 0\}$ and the range is $\{y|y \in R\}$.

(e) $\{(x,y)|x^2 - 4y^2 = -16\}$

(i) **hyperbola**

(ii) x-intercepts

Let $y = 0$.

$x^2 - 4(0)^2 = -16$

$x^2 = -16$

$x = \pm\sqrt{-16}$

no real value

no x-intercepts

(iii) y-intercepts

Let $x = 0$.

$0^2 - 4y^2 = -16$

$-4y^2 = -16$

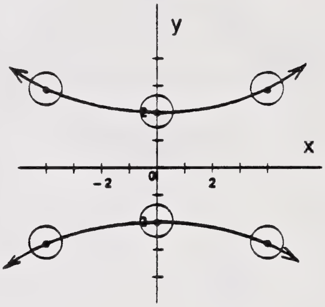
$y^2 = 4$

$y = \pm 2$

y-intercepts are -2 and 2.

(iv) Graph

x	0	0	±4
y	2	−2	±2.8



(v) The domain is $\{x|x \in \mathbf{R}\}$ and the range is $\{y|y \leq -2 \text{ or } y \geq 2\}$.

(f) $\{(x,y)|4x^2 + y^2 = 16\}$

(i) **ellipse**

(ii) x-intercepts

Let $y = 0$.

$4x^2 + 0^2 = 16$

$4x^2 = 16$

$x^2 = 4$

$x = \pm 2$

x-intercepts are -2 and 2.

(iii) y-intercepts

Let $x = 0$.

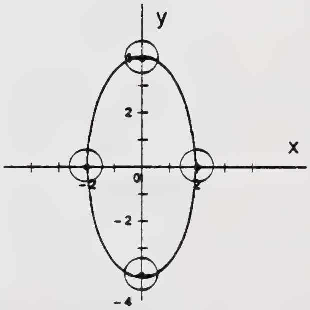
$4(0)^2 + y^2 = 16$

$y^2 = 16$

$y = \pm 4$

y-intercepts are -4 and 4.

(iv) Graph



(v) The domain is $\{x|-2 \leq x \leq 2\}$ and the range is $\{y|-4 \leq y \leq 4\}$.

(g) $\{(x,y) | x^2 - y^2 = 25\}$

(i) **hyperbola**

(ii) x-intercepts

Let $y = 0$.

$$x^2 - (0)^2 = 25$$

$$x^2 = 25$$

$$x = \pm 5$$

x-intercepts are -5 and 5.

(iii) y-intercepts

Let $x = 0$.

$$0^2 - y^2 = 25$$

$$y^2 = -25$$

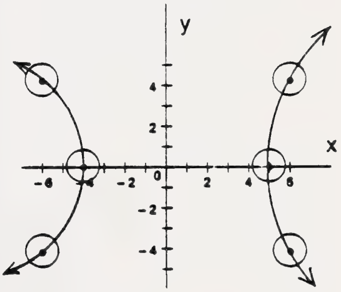
$$y = \pm\sqrt{-25}$$

no real value

no y-intercepts

(iv) Graph

x	±5	±6
y	0	±3.3



(v) The domain is $\{x | x \leq -5 \text{ or } x \geq 5\}$ and the range is $\{y | y \in \mathbf{R}\}$.

3. (a) false (b) true (c) false (d) false
(e) false (f) true (g) true (h) true

4. (a) $\{(x,y) | y = x^2 - 2x - 5\}$

(i) **upward**

(ii)
$$\frac{-b}{2a} = \frac{-(-2)}{2(1)} = \frac{2}{2} = 1$$

$$\frac{4ac - b^2}{4a} = \frac{4(1)(-5) - (-2)^2}{4(1)} = \frac{-20 - 4}{4} = \frac{-24}{4} = -6$$

The vertex occurs at $(1, -6)$.

(iii) The equation of the axis of symmetry is $x = 1$.

(iv) The domain is **R**.

The range is $\{y | y \geq -6\}$.

(b) $\{(x,y)|x = -5y^2 + 4y - 2\}$

(i) to the left

(ii) $\frac{-b}{2a} = \frac{-4}{2(-5)} = \frac{-4}{-10} = \frac{2}{5}$

$\frac{4ac - b^2}{4a} = \frac{4(-5)(-2) - (4)^2}{4(-5)} = \frac{40 - 16}{-20} = \frac{24}{-20} = \frac{-6}{5}$

The vertex occurs at $\left(\frac{-6}{5}, \frac{2}{5}\right)$.

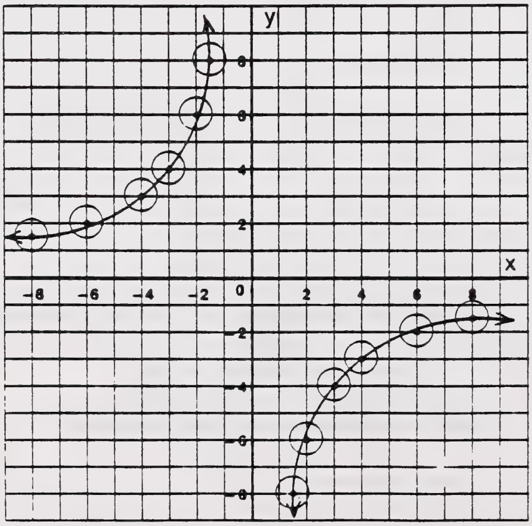
(iii) The equation of the axis of symmetry is $y = \frac{2}{5}$.

(iv) The domain is $\left\{x|x \le \frac{-6}{5}\right\}$.

The range is **R**.

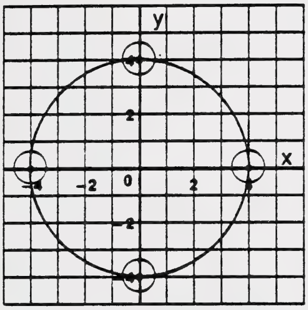
5. (a) $\{(x,y)|xy = -12\}$

x	−8	−6	−4	−3	−2	2	3	4	6	8
y	$1\frac{1}{2}$	2	3	4	6	−6	−4	−3	−2	$-1\frac{1}{2}$



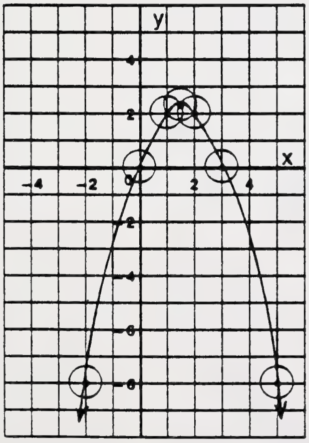
(b) $\{(x,y) | x^2 + y^2 = 16\}$

x	±4	0
y	0	±4



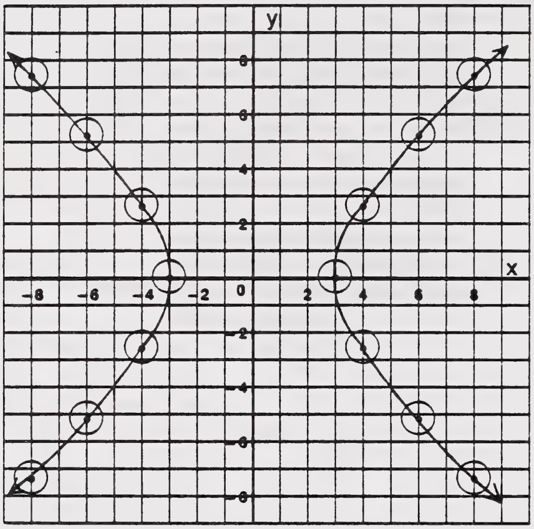
(c) $\{(x,y) | y = -x^2 + 3x\}$

x	−2	−1	0	1	$1\frac{1}{2}$	2	3	4	5
y	−10	−4	0	2	$2\frac{1}{4}$	2	0	−4	−10



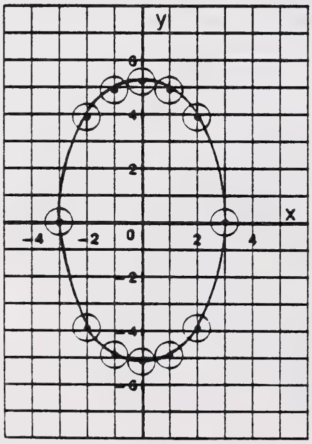
(d) $\{(x,y)|x^2 - y^2 = 9\}$

x	±3	±4	±6	±8
y	0	±2.6	±5.2	±7.4



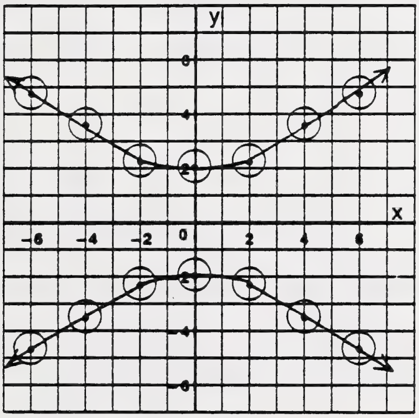
(e) $\{(x,y)|3x^2 + y^2 = 27\}$

x	0	±1	±2	±3
y	±5.2	±4.9	±3.9	0



(f) $\{(x,y) | x^2 - 2y^2 = -8\}$

x	0	±2	±4	±6
y	±2	±2.4	±3.5	±4.7



END OF LESSON H

LESSON I

Page 6

	Think Step	Simplification
$-12i + 11i$	$(-12 + 11)i$	$-i$
$7i + i$	$(7 + 1)i$	$8i$
$8i - 3i$	$(8 - 3)i$	$5i$
$5i + 4i - 7i$	$(5 + 4 - 7)i$	$2i$
$-8i + 2i - 3i$	$(-8 + 2 - 3)i$	$-9i$
$5i - 6i + i$	$(5 - 6 + 1)i$	$0i$
$3i - 5i + 6i - 8i$	$(3 - 5 + 6 - 8)i$	$-4i$
$2i - i + 3i - 6i$	$(2 - 1 + 3 - 6)i$	$-2i$

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1. $-2 + 7i$
2. $9 - 5i$
3. $(-3 + 2) + (-4i + 7i) = -1 + 3i$
4. $5 - 4i$
5. $-6 + (-5i + 8i) = -6 + 3i$
6. $(-8 + 3) - 2i = -5 - 2i$
7. $(-2 + 7) + (8i - 9i) = 5 - i$

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1. $3 - 7i$
2. $-4 - i$
3. $(3 + 2i) + (4 + i) = (3 + 4) + (2i + i) = 7 + 3i$
4. $6i + (-4 - 8i) = -4 + (6i - 8i) = -4 - 2i)$
5. $(3 - 4i) + (-5 + 7i) = (3 - 5) + (-4i + 7i) = -2 + 3i$

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1. -14
2. $(6 \times 3)(i \times i) = 18i^2 = 18(-1) = -18$
3. $(3 \times 1)(i \times i) = 3i^2 = 3(-1) = -3$
4. $2i \times -8i = (2 \times -8)(i \times i) = -16i^2 = -16(-1) = +16$
5. $(-7 \times -1)(i \times i) = 7i^2 = 7(-1) = -7$

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1. $-15i + 21i^2 = -15i - 21$
2. $-4(1) + (-4)(i) = -4 - 4i$
3. $(-2i)(-3) + (-2i)(5i) = 6i - 10i^2 = 6i + 10$
4. $\frac{1}{2}(4) + \frac{1}{2}(-6i) = 2 - 3i$
5. $i - i^2 = i - (-1) = i + 1$

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1. $-2 - 23i$
2. $2 + 2i + 3i + 3i^2 = 2 + 5i - 3 = -1 + 5i$
3. $6 - 10i - 6i + 10i^2 = 6 - 16i - 10 = -4 - 16i$
4. $-3 - 6i - i - 2i^2 = -3 - 7i + 2 = -1 - 7i$
5. $8 - 10i + 4i - 5i^2 = 8 - 6i + 5 = 13 - 6i$

Page 14

1. $25 + 16 = 41$
2. $36 + 1 = 37$
3. $4 - 64i^2 = 4 + 64 = 68$
4. $4 - 25i^2 = 4 + 25 = 29$
5. $1 - i^2 = 1 + 1 = 2$

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1. $\frac{2i}{i} = 2$

2. $\frac{-15}{3} \times \frac{i}{i} = -5$

3. $\frac{\frac{1}{2}}{4} \times \frac{i}{i} = \frac{1}{8}$

4. $\frac{-12}{-4} \times \frac{i}{i} = 3$

5. $\frac{1}{5} \times \frac{i}{i} = \frac{1}{5}$

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1. $-5i$

2. $\frac{-i}{2i^2} = \frac{-i}{-2} = \frac{i}{2}$

3. $\frac{4}{5i} \times \frac{i}{i} = \frac{4i}{5i^2} = \frac{4i}{5(-1)} = \frac{4i}{-5} = \frac{-4i}{5}$

4. $\frac{-3}{7i} \times \frac{i}{i} = \frac{-3i}{7i^2} = \frac{-3i}{7(-1)} = \frac{-3i}{-7} = \frac{3i}{7}$

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1. $\frac{-18 - 27i + 10}{36 + 25}$

$= \frac{-8 - 27i}{61}$

$= \frac{-8}{61} - \frac{27}{61}i$

2. $\frac{2 + 3i}{5 + 2i} \times \frac{5 - 2i}{5 - 2i}$

$= \frac{10 + 15i - 4i - 6i^2}{25 - 4i^2}$

$= \frac{10 + 11i + 6}{25 + 4}$

$= \frac{16 + 11i}{29}$

$= \frac{16}{29} + \frac{11}{29}i$

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$i^9 = i$

$i^{12} = (-1)^6 \times 1 = 1$

$i^{23} = (i^2)^{11} \times i = (-1)^{11} \times i = -1i$

$i^{14} = (i^2)^7 = (-1)^7 = -1$

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2. product, complex

$2i \times 3i = 6i^2 = -6$	$4(2 + i) = 8 + 4i$
$(3 + i)(2 - i) = 7 - i$	$3 \times 5i = 15i$
	Yes

3. add

$5 + 9i \mid 5 + 9i$

4. order, multiply

$6 + 2i - 15i - 5i^2$	$6 - 15i + 2i - 5i^2$
$= 6 - 13i + 5$	$= 6 - 13i + 5$
$= 11 - 13i$	$= 11 - 13i$

5.

$3i + (7 + 4i)$	$[(3i + 4i) + 2] + 5$
$= 3i + 7 + 4i$	$= 7i + 2 + 5$
$= 7 + 7i$	$= 7 + 7i$

6. product

$i(6 + 3i - 2i - i^2)$	$[i(2 + i)](3 - i)$
$= i(6 + i + 1)$	$= (2i + i^2)(3 - i)$
$= i(7 + i)$	$= (2i - 1)(3 - i)$
$= 7i + i^2$	$= 6i - 3 - 2i^2 + i$
$= 7i - 1$	$= 7i - 3 + 2$
	$= 7i - 1$

7.

$-2i(7 - 2i)$	$-6i - 2i^2 - 8i + 6i^2$
$= -14i + 4i^2$	$= -14i + 2 - 6$
$= -14i - 4$	$= -14i - 4$

8. $(3 + 2i) + 0 = 3 + 2i$, $-5 + 0 = -5$, $15i + 0 = 15i$ Yes

9. Multiplied, unchanged

   $1 \times (3 + 2i) = 3 + 2i$, $1 \times 4i = 4i$, $1 \times (-2 + 5i) = -2 + 5i$ Yes

10. (ii) $2 - 5i$ and $-2 + 5i$, $2 - 5i$ (iii) $-3i$, $-3i$

11. multiplication

    (ii) $\frac{1}{1 + 5i}$, 1 (iii) $\frac{1}{2i}$, $\frac{1}{2i}$

Pages 28-33

Exercise – The Complex Number System

1. (a) rational (b) additive
   (c) multiplicative (d) pure imaginary , imaginaries
   (e) $a + bi$, real (f) subset
   (g) $\sqrt{-1}$, -1 (h) 2 , -5
   (i) real (j) conjugate or multiplicative inverse
   (k) $(-a + bi)$ (l) complex

2.

$4 + 3i$, $-7 + 2i$ $= (4 + 3i) + (-7 + 2i)$ $= (4 - 7) + (3i + 2i)$ $= -3 + 5i$	$6 - 3i$, $-2 - 5i$ $= (6 - 3i) + (-2 - 5i)$ $= (6 - 2) + (-3i - 5i)$ $= 4 - 8i$	5 , $-9 + 4i$ $= 5 + (-9 + 4i)$ $= (5 - 9) + 4i$ $= -4 + 4i$
$3i$, $-7 + 8i$ $= 3i + (-7 + 8i)$ $= -7 + (3i + 8i)$ $-7 + 11i$	$3 + 6i$, $4 - i$ $= (3 + 6i) + (4 - i)$ $= (3 + 4) + (6i - i)$ $= 7 + 5i$	$-3 - i$, $5 + i$ $= (-3 - i) + (5 + i)$ $= (-3 + 5) + (-i + i)$ $= -2 + 0i$ $= -2$

3.

$2 + 3i\ ,\ -5 + 7i$ $= (2 + 3i) - (-5 + 7i)$ $= (2 + 3i) + (5 - 7i)$ $= (2 + 5) + (3i - 7i)$ $= 7 - 4i$	$6 + 4i\ ,\ 3 - i$ $= (6 + 4i) - (3 - i)$ $= 6 + 4i - 3 + i$ $= (6 - 3) + (4i + i)$ $= 3 + 5i$	$-2i\ ,\ 5 + 4i$ $= -2i - (5 + 4i)$ $= -2i - 5 - 4i$ $= -5 + (-2i - 4i)$ $= -5 - 6i$
$\sqrt{2} + 3i\ ,\ 5\sqrt{2} - i$ $= (\sqrt{2} + 3i) - (5\sqrt{2} - i)$ $= \sqrt{2} + 3i - 5\sqrt{2} + i$ $= \sqrt{2} - 5\sqrt{2} + 3i + i$ $= -4\sqrt{2} + 4i$	$2 - i\ ,\ 3$ $= (2 - i) - 3$ $= 2 - 3 - i$ $= -1 - i$	$5 + 4i\ ,\ -3 - 8i$ $= (5 + 4i) - (-3 - 8i)$ $= 5 + 4i + 3 + 8i$ $= 8 + 12i$

4. (a) $-6i^2 = -6(-1) = 6$

(b) $7i^2 = 7(-1) = -7$

(c) $-12i$

(d) $28 + 21i + 8i + 6i^2$

$= 28 + 29i - 6$

$= 22 + 29i$

(e) $-12 + 6i + 24i - 12i^2$

$= -12 + 30i + 12$

$= 30i$

(f) $4 - 25i^2 = 4 + 25 = 29$

(g) $-8i + 6i^2 = -8i - 6$

(h) $-10 + 30i$

(i) $6 - 2i - 15i + 5i^2$

$= 6 - 17i - 5$

$= 1 - 17i$

(j) $8 - 3\sqrt{2}i + 8\sqrt{2}i - 6i^2$

$= 8 + 5\sqrt{2}i + 6$

$= 14 + 5\sqrt{2}i$

5. (a)

$$= \frac{-2}{i} \times \frac{i}{i}$$

$$= \frac{-2i}{i^2}$$

$$= \frac{-2i}{-1} = 2i$$

(b)

$$= \frac{6 + 3i}{4 - i^2}$$

$$= \frac{6 + 3i}{4 + 1}$$

$$= \frac{6 + 3i}{5}$$

$$= \frac{6}{5} + \frac{3}{5}i$$

(c)

$$= \frac{6 - 2i}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{6 - 2i - 6i + 2i^2}{1 - i^2}$$

$$= \frac{6 - 8i - 2}{1 + 1} = \frac{4 - 8i}{2}$$

$$= 2 - 4i$$

(d)

$$= \frac{2 + i}{4 - 3i} \times \frac{4 + 3i}{4 + 3i}$$

$$= \frac{8 + 6i + 4i + 3i^2}{16 - 9i^2}$$

$$= \frac{8 + 10i - 3}{16 + 9}$$

$$= \frac{5 + 10i}{25}$$

$$= \frac{1}{5} + \frac{2}{5}i$$

(e)

$$= \frac{3i}{4 + 2i} \times \frac{4 - 2i}{4 - 2i}$$

$$= \frac{12i - 6i^2}{16 - 4i^2}$$

$$= \frac{12i + 6}{16 + 4}$$

$$= \frac{12i + 6}{20}$$

$$= \frac{3}{10} + \frac{3}{5}i$$

6. (a) $x^2 + 5x + 8 = 0$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(8)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{25 - 32}}{2}$$

$$x = \frac{-5 \pm \sqrt{-7}}{2}$$

$$x = \frac{-5 \pm i\sqrt{7}}{2}$$

(b) $2x^2 + 8 = 0$

$$2x^2 = -8$$

$$x^2 = -4$$

$$x = \pm\sqrt{-4}$$

$$x = \pm 2i$$

(c) $6x^2 - x + 2 = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(2)}}{2(6)}$$

$$x = \frac{+1 \pm \sqrt{1 - 48}}{12}$$

$$x = \frac{+1 \pm \sqrt{-47}}{12}$$

$$x = \frac{1 \pm i\sqrt{47}}{12}$$

(d) $3x^2 + 2ix - 1 = 0$

$$x = \frac{-2i \pm \sqrt{(2i)^2 - 4(3)(-1)}}{2(3)}$$

$$x = \frac{-2i \pm \sqrt{4i^2 + 12}}{6}$$

$$x = \frac{-2i \pm \sqrt{-4 + 12}}{6}$$

$$x = \frac{-2i \pm \sqrt{8}}{6}$$

$$= \frac{-2i \pm 2\sqrt{2}}{6}$$

$$= \frac{-i \pm \sqrt{2}}{3}$$

7. (a) $5(3 - 2i)^2$

$$= 5(9 - 12i + 4i^2)$$

$$= 45 - 60i + 20i^2$$

$$= 45 - 60i - 20$$

$$= 25 - 60i$$

(b) $\dfrac{i^3 + i^4}{1 + 2i^2}$

$$= \frac{(i^2)i + (i^2)^2}{1 - 2}$$

$$= \frac{-1i + (-1)^2}{-1}$$

$$= \frac{-i + 1}{-1}$$

$$= i - 1$$

(c) $2i(1 + i) + 4(3 - 2i)$

$= 2i + 2i^2 + 12 - 8i$

$= 2i - 2 + 12 - 8i$

$= +10 - 6i$

(d) $(2 + i)(2 - i) - (5 - i)$

$= 4 - i^2 - 5 + i$

$= 4 + 1 - 5 + i$

$= i$

(e) $-5(1 - i) + 5 + 5i^3$

$= -5 + 5i + 5 + 5i(i^2)$

$= 5i - 5i$

$= 0$

(f) $i^{24} \times (2i)^5$

$= (i^2)^{12} \times 32i^5$

$= (-1)^{12} \times 32(i^2)^2 i$

$= 1 \times 32(-1)^2 i$

$= 32i$

(g) $(3 + 2i)(4 - i) - (5 + i)(7 - i)$

$= (12 + 8i - 3i - 2i^2) - (35 + 7i - 5i - i^2)$

$= (12 + 5i + 2) - (35 + 2i + 1)$

$= (14 + 5i) - (36 + 2i)$

$= 14 + 5i - 36 - 2i$

$= -22 + 3i$

(h) $(3 + i)^4$

$= (3 + i)^2(3 + i)^2$

$= (9 + 6i + i^2)(9 + 6i + i^2)$

$= (9 + 6i - 1)(9 + 6i - 1)$

$= (8 + 6i)(8 + 6i)$

$= 64 + 48i + 48i + 36i^2$

$= 64 + 96i - 36$

$= 28 + 96i$

8. $(2 + i)^2 - 4(2 + i)$

$= 4 + 4i + i^2 - 8 - 4i$

$= 4 + 4i - 1 - 8 - 4i$

$= -5$

9. $2\left(\frac{1 + i}{2}\right)^2 - \overset{1}{\cancel{2}}\left(\frac{1 + i}{\cancel{2}_1}\right) + 1$

$= \overset{1}{\cancel{2}}\left(\frac{1 + 2i + i^2}{\cancel{4}_2}\right) - 1 - i + 1$

$= \frac{1 + 2i - 1}{2} - i$

$= \frac{2i}{2} - i$

$= i - i$

$= 0$

10. (a) true (b) false (c) false
(d) true (e) false (f) true
(g) true (h) true (i) false

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$2x - y = 3$ ①

$-y = 3 - 2x$

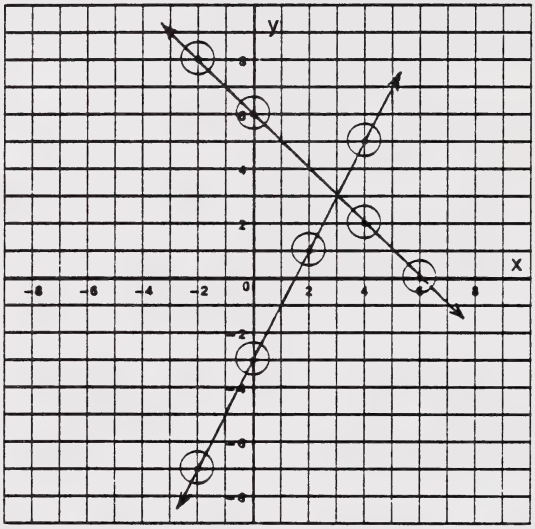
$y = 2x - 3$

x	−2	0	2	4
y	−7	−3	1	5

$x + y = 6$ ②

$y = 6 - x$

x	−2	0	4	6
y	8	6	2	0



The two straight lines intersect at the point whose coordinates are (3,3).

Thus, the solution set of the system is $\{(3,3)\}$.

This means that both equations are satisfied when $x = 3$ and $y = 3$.

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Elimination Method

$x - y = 1$ ①

$\underline{2x + y = -4}$ ②

$3x = -3$ (add)

$x = -1$

Substitute $x = -1$ in ①

$-1 - y = 1$

$-y = 2$

$y = -2$

The solution set is $\{(-1,-2)\}$.

Substitution Method

$x - y = 1$ ①.

$2x + y = -4$ ②

From equation ①, $x = y + 1$.

Substitute this value for x in ②.

$$2(y + 1) + y = -4$$
$$2y + 2 + y = -4$$
$$3y = -6$$
$$y = -2$$

Substitute $y = -2$ into ①.

$$x - (-2) = 1$$
$$x = -1$$

The solution set is $\{(-1,-2)\}$.

x	−2	0	2
y	−3	−1	1

x	−2	0	2
y	0	2	4

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Elimination Method

$x - y = 1$ ①

$\underline{x - y = -2}$ ②

$0 = 3$ (subtract)

This statement is always false.

The solution set is ϕ.

Substitution Method

$x - y = 1$ ①

$x - y = -2$ ②

From equation ①, $x = y + 1$.

Substitute this value of x in ②.

$$(y + 1) - y = -2$$
$$1 = -2$$

This statement is always false.

The solution set is ϕ.

x	−3	1	3
y	3	1	0

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Elimination Method	Substitution Method
$x + 2y = 3$ ① $2x + 4y = 6$ ②	$x + 2y = 3$ ① $2x + 4y = 6$ ②
	From equation ①, $x = 3 - 2y$. Substitute this value for x in ②.
$2x + 4y = 6$ ① × 2 $\underline{2x + 4y = 6}$ ② $0 = 0$ (subtract)	$2(3 - 2y) + 4y = 6$ $6y - 4y + 4y = 6$ $6 = 6$
This statement is always true. The solution set contains an infinite number of ordered pairs.	This statement is always true. The solution set is an infinite number of ordered pairs.

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Example 1: straight line

x	−1	2	4
y	8	5	3

hyperbola

x	−6	−4	−3	−2	2	3	4	6
y	−2	−3	−4	−6	6	4	3	2

(4,3) , (4,3)

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Example 2: straight line $y = \dfrac{-4x + 25}{3}$

x	1	4	7
y	7	3	−1

circle , 5

Example 3: straight line $y = -2x - 8$

x	−5	−3	−1
y	2	−2	−6

parabola , upward

x	−3	−2	−1	0	1	2	3	4
y	6	0	−4	−6	−6	−4	0	6

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Example 1: $7x - x^2 = 12$

$$x^2 - 7x + 12 = 0$$

$$(x - 4)(x - 3) = 0$$

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Example 2: $3y = -4x + 25$

$$y = \frac{-4x + 25}{3}$$

We can substitute $\frac{-4x + 25}{3}$ for y.

$$9x^2 + 16x^2 - 200x + 625 = 225$$

$$25x^2 - 200x + 400 = 0$$

$$x^2 - 8x + 16 = 0$$

$$(x - 4)(x - 4) = 0$$

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$y = -2x - 8$

We can substitute $-2x - 8$.

$x^2 + x + 2 = 0$

In this case, $a = 1$, $b = 1$, $c = 2$

$$x = \frac{-1 \pm \sqrt{1 - 8}}{2} = \frac{-1 \pm \sqrt{-7}}{2}$$

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1. (a) $x + y = 0$ ①

$2x + 3y = 3$ ②

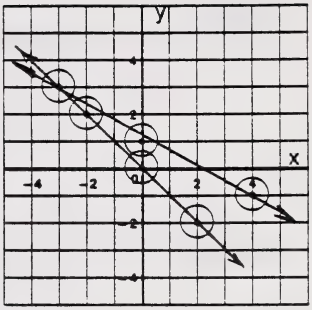
$y = -x$ ①

x	−2	0	2
y	2	0	−2

$y = \dfrac{-2x + 3}{3}$ ②

x	−3	0	3
y	3	1	−1

The system is **independent.**



The solution set is $\{(-3,3)\}$.

(b) $2y = x + 1$ ①

$2x + 2 = 4y$ ②

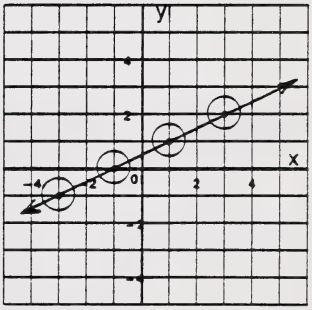
$y = \dfrac{x + 1}{2}$ ①

x	−3	−1	1
y	−1	0	1

$y = \dfrac{x + 1}{2}$ ②

x	−3	1	3
y	−1	1	2

The system is **dependent.**



The solution set is **an infinite set of ordered pairs.**

(c) $x - 2y = 0$ ①

$2y - x = 2$ ②

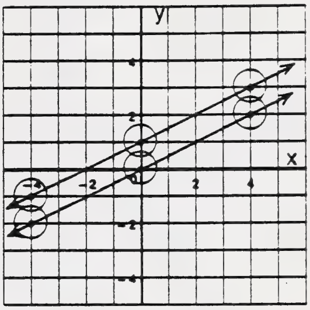
$y = \frac{x}{2}$

x	−4	0	4
y	−2	0	2

$y = \frac{x + 2}{2}$

x	−4	0	4
y	−1	1	3

The system is **inconsistent.**



The solution set is ϕ.

(d) $3y = x - 4$ ①

$2x - 1 = y + 7$ ②

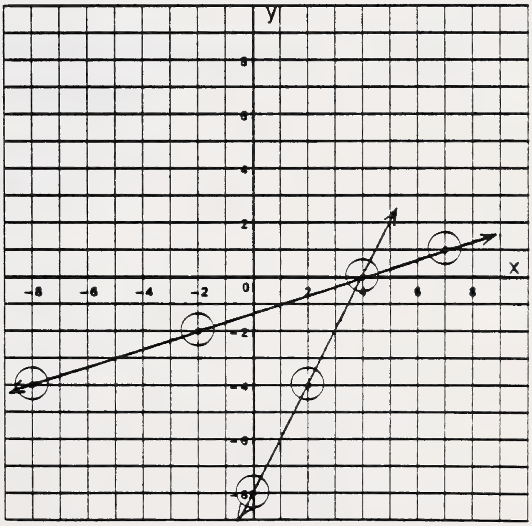
$y = \frac{x - 4}{3}$

x	−8	−2	7
y	−4	−2	1

$y = 2x - 8$

x	0	2	4
y	−8	−4	0

The system is **independent.**



The solution set is {(4,0)}.

2. (a) $$\begin{cases} 3x + 4y = 2 & \text{①} \\ 2x - y = -6 & \text{②} \times 4 \end{cases}$$

$$\begin{array}{r} \left\{\begin{array}{r} 3x + 4y = 2 \\ 8x - 4y = -24 \end{array}\right. \\ \hline 11x = -22 \end{array}$$

$$x = -2$$

Substitute $x = -2$ into ①.

$$\begin{aligned} 3(-2) + 4y &= 2 \\ 4y &= 2 + 6 \\ 4y &= 8 \\ y &= 2 \end{aligned}$$

Solution set is $\{(-2,2)\}$.

(b) $$\begin{cases} 2x - 3y = -24 & \text{①} \times 3 \\ 3x - 5y = 22 & \text{②} \times -2 \end{cases}$$

$$\begin{array}{r} \left\{\begin{array}{r} 6x - 9y = -72 \\ -6x + 10y = -44 \end{array}\right. \\ \hline y = -116 \end{array}$$

Substitute $y = -116$ into ①.

$$\begin{aligned} 2x - 3(-116) &= -24 \\ 2x + 348 &= -24 \\ 2x &= -372 \\ x &= -186 \end{aligned}$$

Solution set is $\{(-186,-116)\}$.

3. (a) $$\begin{cases} 3x + 6y = 13 & \text{①} \\ 9x + y = 5 & \text{②} \end{cases}$$

From ② we get $y = -9x + 5$.

Substitute this into ①.

$$\begin{aligned} 3x + 6(-9x + 5) &= 13 \\ 3x - 54x + 30 &= 13 \\ -51x &= -17 \\ x &= \frac{-17}{-51} \\ &= \frac{1}{3} \\ y &= -9x + 5 \\ &= -9\left(\frac{1}{3}\right) + 5 \\ &= -3 + 5 \\ &= 2 \end{aligned}$$

Solution set is $\left\{\left(\frac{1}{3},2\right)\right\}$.

(b) $$\begin{cases} 2x - y = 7 & \text{①} \\ 3x + 2y = 14 & \text{②} \end{cases}$$

From ① we get $y = 2x - 7$.

Substitute this into ②.

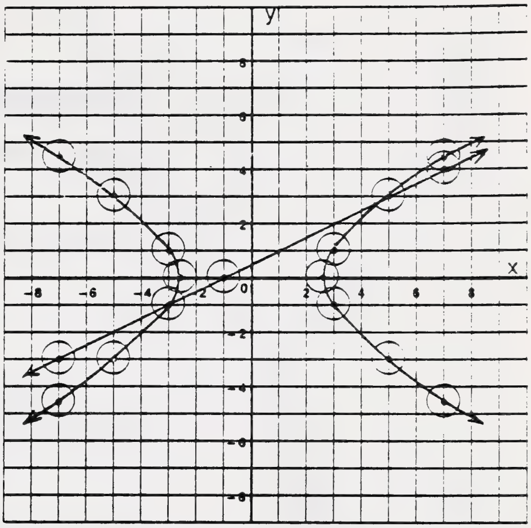
$$\begin{aligned} 3x + 2(2x - 7) &= 14 \\ 3x + 4x - 14 &= 14 \\ 7x &= 28 \\ x &= 4 \end{aligned}$$

$$\begin{aligned} y &= 2(4) - 7 \\ &= 1 \end{aligned}$$

Solution set is $\{(4,1)\}$.

4. (a) $x^2 - 2y^2 = 7$ ①

$2y - x = 1$ ②



Graphical Solution

The graph of equation ① is a **hyperbola**.

$$y = \pm\sqrt{\frac{x^2 - 7}{2}}$$

x	±2.6	±3	±5	±7
y	0	±1	±3	±4.6

The graph of equation ② is a **straight line**.

$$y = \frac{x + 1}{2}$$

x	−7	−1	7
y	−3	0	4

The graphs have **2** points in common. The solution set is $\{(-3,-1),(5,3)\}$.

$x^2 - 2y^2 = 7$ ①

$2y - x = 1$ ②

Algebraic Solution

From equation ②,
$$\begin{aligned} 2y - x &= 1 \\ -x &= -2y + 1 \\ \mathbf{x} &= \mathbf{2y - 1} \end{aligned}$$

Substitute **2y − 1** for x in equation ①.
$$\begin{aligned} (2y - 1)^2 - 2y^2 &= 7 \\ 4y^2 - 4y + 1 - 2y^2 &= 7 \\ 2y^2 - 4y - 6 &= 0 \\ y^2 - 2y - 3 &= 0 \\ (y - 3)(y + 1) &= 0 \end{aligned}$$

When $y = 3$, $x = 2y - 1 = 2(3) - 1 = 5$

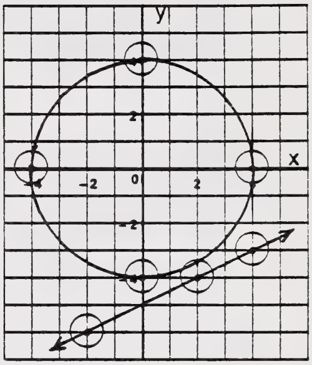
When $y = -1$, $x = 2y - 1 = 2(-1) - 1 = -3$

The solution set is $\{(5,3),(-3,-1)\}$.

(b) $x^2 + y^2 = 16$ ①

$2y = x - 10$ ②

Graphical Solution



The graph of equation ① is a **circle** with a radius of **4**.

The graph of equation ② is a **straight line**.

$$y = \frac{x - 10}{2}$$

x	−2	2	4
y	−6	−4	−3

The two graphs have **no** points in common.

The solution set is ϕ.

$x^2 + y^2 = 16$ ①

$2y = x - 10$ ②

Algebraic Solution

From equation ②, $x = 2y + 10$.

Substitute **2y + 10** for x in equation ①.

$$(2y + 10)^2 + y^2 = 16$$

$$4y^2 + 40y + 100 + y^2 = 16$$

$$5y^2 + 40y + 84 = 0$$

$$y = \frac{-40 \pm \sqrt{40^2 - 4(5)(84)}}{2(5)}$$

$$y = \frac{-40 \pm \sqrt{1600 - 1680}}{10}$$

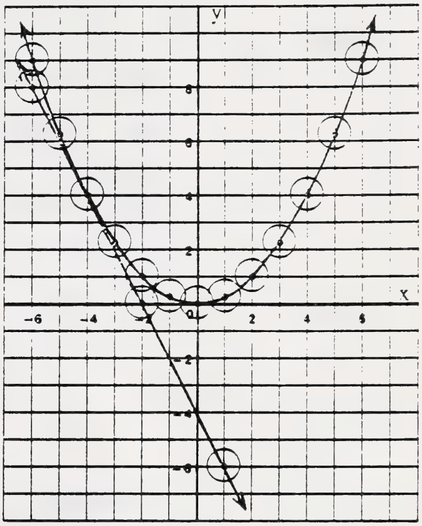
$$y = \frac{-40 \pm \sqrt{-80}}{10}$$

There are **no** real roots.

The solution set is ϕ if $x, y \in R$.

(c) $2x + y + 4 = 0$ ①

$4y = x^2$ ②



Graphical Solution

The graph of equation ① is a **straight line.**

$y = -2x - 4$

x	−6	−2	1
y	8	0	−6

The graph of equation ② is a **parabola** which opens **upward.**

$$y = \frac{x^2}{4}$$

x	0	±1	±2	±3	±4	±5	±6
y	0	$\frac{1}{4}$	1	2.25	4	$6\frac{1}{4}$	9

The two graphs have **one** point in common.

The solution set is $\{(-4,4)\}$.

$2x + y + 4 = 0$ ①

$4y = x^2$ ②

Algebraic Solution

From equation ①, $y = -2x - 4$.

Substitute $-2x - 4$ for y in equation ②.

$$4(-2x - 4) = x^2$$
$$-8x - 16 = x^2$$
$$x^2 + 8x + 16 = 0$$
$$(x + 4)(x + 4) = 0$$
$$x = -4$$

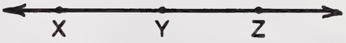
When $x = -4$, $y = -2(-4) - 4 = 8 - 4 = 4$

The solution set is $\{(-4,4)\}$.

END OF LESSON I

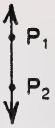
LESSON J

Page 2



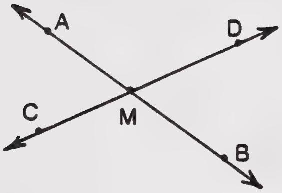
$\overleftrightarrow{YZ}$, $\overleftrightarrow{XZ}$, $\overleftrightarrow{ZX}$, $\overleftrightarrow{XY}$, $\overleftrightarrow{YX}$, $\overleftrightarrow{ZY}$

(one only needed)



$\overleftrightarrow{P_1P_2}$ or $\overleftrightarrow{P_2P_1}$

Page 3



non-collinear , collinear

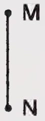
Page 4

P_1, P_2, P_4 or any other three points

P_1, P_3, P_4, P_5

Yes

Page 5



$\overline{MN}$ or $\overline{NM}$

$\overline{PS}$ or $\overline{SP}$

Page 6

AC = 3.4 cm MN = 1.5 cm PS = 4.7 cm

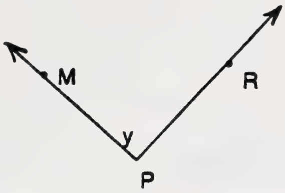
Page 7

$\overline{PQ}$ $\overrightarrow{ZX}$ or $\overrightarrow{ZY}$

Page 8

Rays: $\overrightarrow{YX}$, $\overrightarrow{YZ}$ Rays: $\overrightarrow{TR}$, $\overrightarrow{TV}$ Rays: O[illegible] , [illegible]

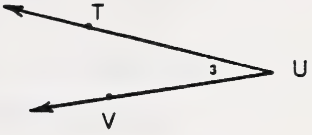
Vertex: Y Vertex: T Vertex: O



∠P

∠y

∠MPR



∠U

∠3

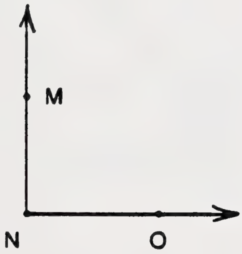
∠TUV

Page 9

m∠MNO = 120° , m∠PQR = 45° , m∠DEF = 70°

Page 10

1.



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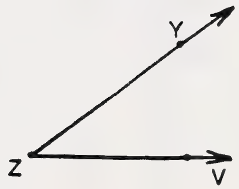
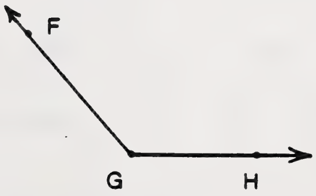


3.



4. $\overrightarrow{YX}$, $\overrightarrow{YZ}$

Page 11

$\overline{CD}$ and $\overline{DE}$

Page 12

The least number of sides a polygon can have is 3.

This polygon has 5 sides.

There are 5 vertices.

Its vertices are M, N, O, P, Q, R.

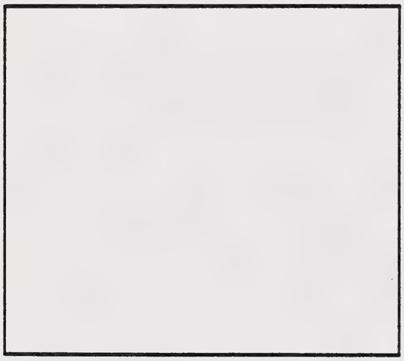
Its sides are $\overline{MN}$, $\overline{NO}$, $\overline{OP}$, $\overline{PQ}$, $\overline{RQ}$, $\overline{MR}$

The diagonals are: $\overline{AC}$, $\overline{CE}$, $\overline{BE}$, $\overline{BD}$

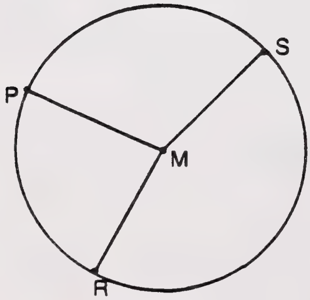
Page 13

no , yes , no

Page 14



Page 15

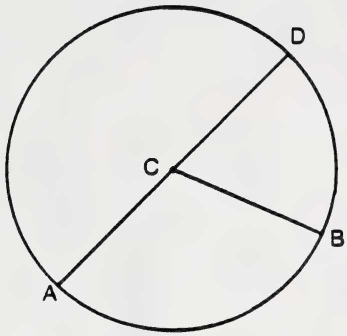


Name of circle: **Circle M**

Three points: **P, R, S**

Three radii: **$\overline{MP}$, $\overline{MS}$, $\overline{MR}$**

Measure of radii: **2.2 cm**

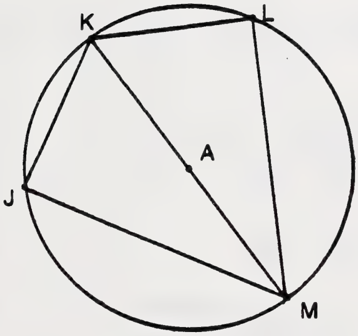


Measure of $\overline{CB}$: **2.5 cm**

Measure of $\overline{AD}$: **5.1 cm**

Yes

Page 16



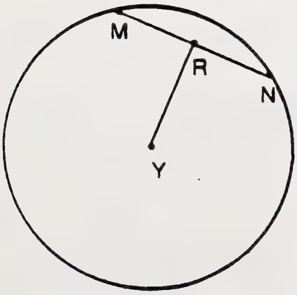
Two pairs of congruent chords:

$\overline{JK} \cong \overline{KL}$

$\overline{JL} \cong \overline{JM}$

or $\overline{LM}$

$\overline{KM}$



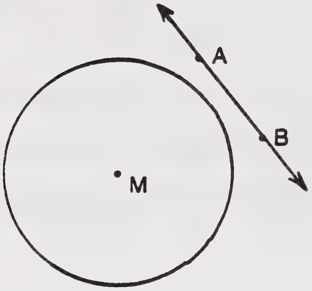
m∠YRN = **90°**

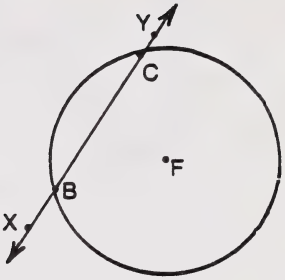
right angle

Point of tangency: **F**

m∠OFG = 90°

∠OFG is a **right** angle.





The greatest number of points that a line can have in common with a circle is **2**.

Pages 20-26

Exercise – Basic Geometric Terms

1. (a) segments , rays (b) point (c) lines , rays
(d) non-collinear (e) plane (f) segment
(g) ray (h) rays , endpoint (i) vertex
(j) endpoints (k) protractor (l) obtuse
(m) right (n) coplanar (o) polygon
(p) congruent (q) vertex (r) centre
(s) segment , centre (t) diameter (u) chord
(v) tangent (w) congruent (x) measure
(y) semicircle (z) major

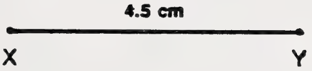
2. (a) $\overrightarrow{CD}$ (b) $\overleftrightarrow{MN}$ (c) $\overline{RS}$
(d) $\angle MNO$ (e) $\overset{\frown}{XYZ}$ (f) $m\angle R$

3.

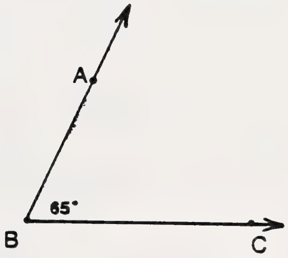
Angle	Measure	Classification
∠1	60°	acute
∠2	30°	acute
∠3	75°	acute
∠DOB	105°	obtuse
∠COA	90°	right
∠DOA	165°	obtuse

4. (a) $\overleftrightarrow{AC}$ or $\overleftrightarrow{AB}$
(b) $\overline{AB}$, $\overline{AC}$, $\overline{BC}$
(c) $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{BD}$
(d) ∠DBC , ∠ABC , ∠ABD
(e) A , B , C
(f) C ; D , B
(g) B
(h) $\overrightarrow{BD}$, $\overrightarrow{BA}$

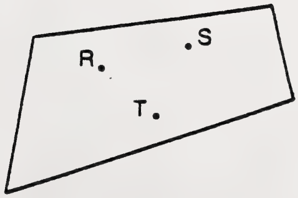
5. (a)



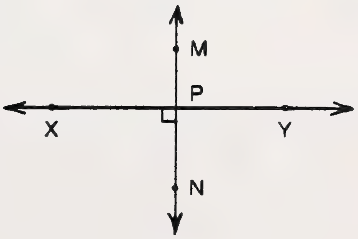
(b)



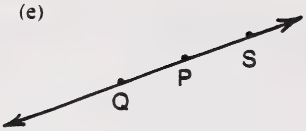
(c)



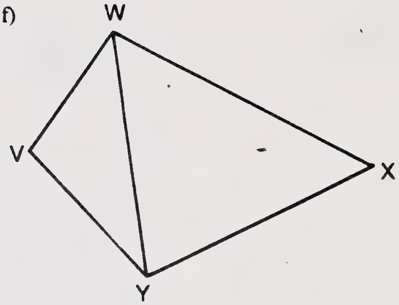
(d)



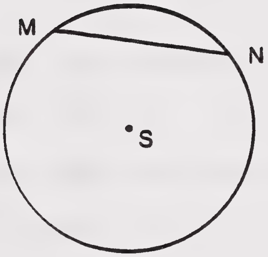
(e)



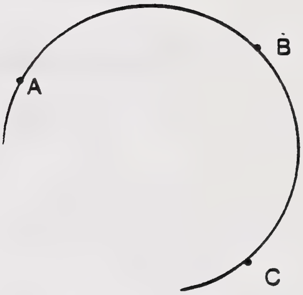
(f)



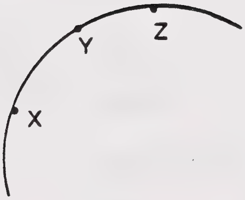
(g)



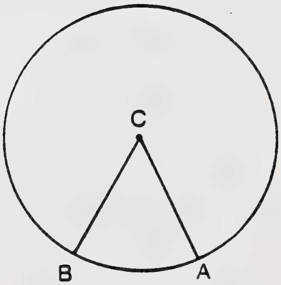
(h)



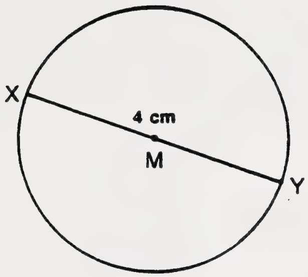
(i)



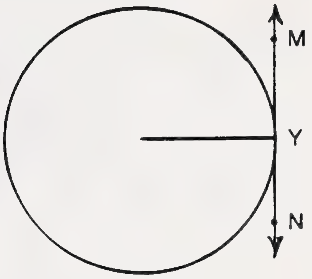
(j)



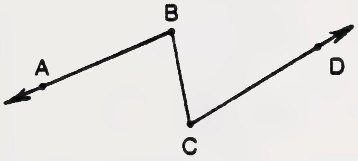
(k)



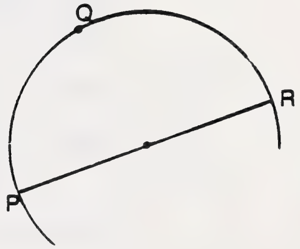
(l)



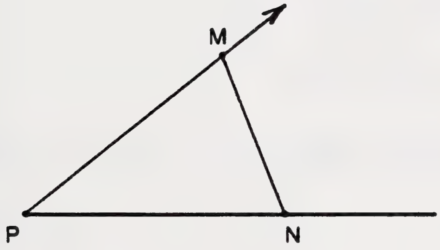
(m)



(n)



6.



m ∠ PMN = 70°

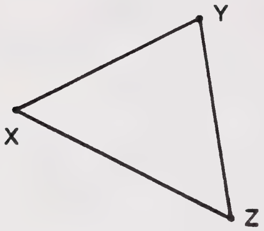
m ∠ PNM = 70°

7. (a) D
(b) C
(c) ∠A, ∠B, ∠C
(d) $\overline{DE}$, $\overline{DF}$, $\overline{EF}$
(e) $\overline{AB}$
(f) $\overline{AD}$, $\overline{DF}$, $\overline{CF}$, $\overline{AC}$

8. (a) Circle P
(b) $\overline{AR}$, 3.9 cm
(c) $\overline{PA}$, $\overline{PS}$, $\overline{PB}$, 1.9 cm
(d) $\overline{AB}$, $\overline{RS}$, $\overline{AR}$
(e) A, X, B, S, M, R, Y
(f) $\overset{\frown}{AYR}$, $\overset{\frown}{ABR}$
(g) ∠APB, ∠RPS
(h) $\overset{\frown}{AXB}$, $\overset{\frown}{RMS}$
(i) $\overline{AB}$, $\overline{RS}$
(j) $\overleftrightarrow{AC}$, A, $\overline{PA}$
(k) $\overrightarrow{AR}$, 2

Page 27

The three vertices of this triangle are: **X**, **Y**, and **Z**.



Side	Measure
$\overline{PR}$	5.8 cm
$\overline{PQ}$	4.2 cm
$\overline{QR}$	5.6 cm

Angle	Measure
∠R	43°
∠P	67°
∠Q	70°

The sum of the measures of the 3 angles of ΔPQR is **180°**.

Page 28

1. AB = AC = 3.8 cm

   m∠B = m∠C = 70°

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2. $\overline{MN} \cong \overline{MO} \cong \overline{NO}$ since MN = MO = NO = 3.3 cm

   ∠M ≅ ∠N ≅ ∠O since m∠M = m∠N = m∠O = 60°

3. i.e. QP = 5.0 c. , QR = 2.6 cm and PR = 3.9 cm

1. m∠D = 50° , m∠E = 70°

   m∠F = 60°

Page 30

2. m∠H = 130°

3. m∠K = 90° , $\overline{KL}$

4. m∠Z = 60°

Page 31

1. acute
2. equiangular
3. obtuse

1. scalene
2. scalene
3. isosceles
4. equilateral

1. true
2. true
3. false
4. true
5. false
6. true
7. false
8. true
9. true
10. true

Page 37

1. ASA
2. SAS
3. SSS
4. ASA
5. SAS
6. SSS
7. SAS
8. ASA
9. SSS
10. SAS
11. ASA
12. SAS

Page 38

$m\angle N = 70^\circ$

$m\angle M = 60^\circ$

$m\angle O = 50^\circ$

$$\frac{AC}{DF} = \frac{3.8}{1.9} = \frac{2}{1}$$

$$\frac{BC}{EF} = \frac{5}{2.5} = \frac{2}{1}$$

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$$\frac{AC}{DF} = \frac{BC}{EF}, \quad \frac{BC}{EF} = \frac{AB}{DE}, \quad \frac{DF}{AC} = \frac{EF}{BC}$$

$$\frac{PQ}{PR} = \frac{XY}{XZ}, \quad \frac{QR}{PR} = \frac{YZ}{XZ}, \quad \frac{YZ}{XY} = \frac{QR}{PQ}$$

Page 40

$$5x = 56$$

$$x = 11\frac{1}{5}$$

$$\frac{5}{9} = \frac{8}{y}$$

$$5 \times y = 9 \times 8$$

$$5y = 72$$

$$y = \frac{72}{5} = 14\frac{2}{5}$$

Page 43

Triplet (a,b,c,)	a^2	b^2	$a^2 + b^2$	c^2	Does $c^2 = a^2 + b^2$?
(4, 5, 6)	16	25	41	36	no
(12, 15, 18)	144	225	369	324	no
$(1, 2, \sqrt{5})$	1	4	5	5	yes
$(1, \sqrt{3}, 2)$	1	3	4	4	yes
$(\sqrt{3}, \sqrt{4}, \sqrt{5})$	3	4	7	5	no
(3, 4, 5)	9	16	25	25	yes
(9, 12, 15)	81	144	225	225	yes

Page 44

1. $c^2 = a^2 + b^2$
   $c^2 = 5^2 + 6^2$
   $c^2 = 25 + 36$
   $c^2 = 61$
   $c = \sqrt{61}$
   $c = 7.810$

2. $c^2 = a^2 + b^2$
   $6^2 = 3^2 + b^2$
   $b^2 = 36 - 9$
   $b^2 = 27$
   $b = \sqrt{27}$
   $b = 5.196$

3. $c^2 = a^2 + b^2$
   $15^2 = a^2 + 13^2$
   $a^2 = 225 - 169$
   $a^2 = 56$
   $a = \sqrt{56}$
   $a = 7.483$

4. $c^2 = a^2 + b^2$
   $c^2 = 5^2 + 12^2$
   $c^2 = 25 + 144$
   $c^2 = 169$
   $c = \sqrt{169}$
   $c = 13$

Pages 45-49

Exercise – Triangles

1. (a) non-collinear , vertices (b) Congruent , similar

(c) hypotenuse , right , sum (d) B , C

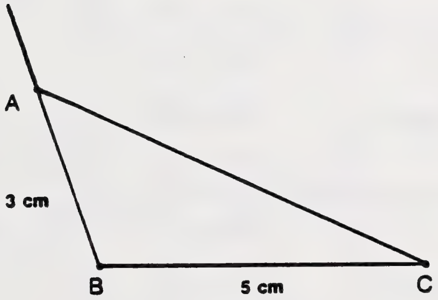
(e) 80°

2 (a) ΔRST (b) $\overline{RT}$, $\overline{ST}$

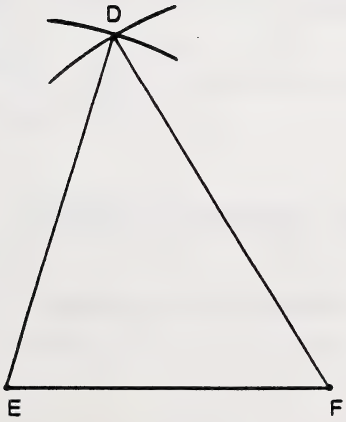
(c) ∠R , ∠S , ∠T (d) R , S , T

(e) isosceles (f) acute

3. (a)



(b)



4.

Property	**Other Congruencies**
ASA Property	$\overline{ED} \cong \overline{AB}$ $\overline{AC} \cong \overline{FD}$ $\angle D \cong \angle A$
SAS Property	$\overline{AE} \cong \overline{CD}$ $\angle E \cong \angle D$ $\angle A \cong \angle C$
SSS Property	$\angle DCA \cong \angle BAC$ $\angle DAC \cong \angle BCA$ $\angle D \cong \angle B$
ASA Property	$\overline{DC} \cong \overline{CB}$ $\overline{DA} \cong \overline{BA}$ $\angle D \cong \angle B$
SAS Property	$\overline{DA} \cong \overline{DC}$ $\angle A \cong \angle C$ $\angle ADB \cong \angle CDB$
ASA Property	$\overline{DC} \cong \overline{AB}$ $\overline{DA} \cong \overline{BC}$ $\angle D \cong \angle B$

5. (a) $\Delta ABD \cong \Delta ACD$ by the SAS property

(b) $\Delta ABC \cong \Delta ADC$ by the ASA property

6. Let one side be x cm.

$$\frac{8}{12} = \frac{10}{x}$$

$$8x = 120$$

$$x = 15$$

Let the longest side by y cm.

$$\frac{8}{12} = \frac{15}{y}$$

$$8y = 15 \times 12$$

$$y = 22\frac{1}{2}$$

The other two sides are 15 cm and $22\frac{1}{2}$ cm.

7. Let x be the height of the flagpole.

$$\frac{x}{15} = \frac{1.8}{2}$$

$$x = 13.5 \text{ m}$$

8. (a) $\frac{CF}{AC} = \frac{5}{12}$

(b) $\frac{DE}{AD} = \frac{5}{12}$

(c) $\frac{DE}{AE}$

(d) $\frac{AC}{AF}, \frac{AD}{AE}$

(e) $AD = 12 + 8 + 8.8 = 28.8$ cm

$$\frac{AD}{DE} = \frac{12}{5}$$

$$\frac{28.8}{DE} = \frac{12}{5}$$

$$DE = 12$$

The length of DE is 12 cm.

9. (a)

$$x^2 = 25 + 36$$

$$x^2 = 61$$

$$x = \sqrt{61}$$

$$x = 7.810$$

(b)

$$5^2 = x^2 + 3^2$$

$$x^2 = 25 - 9$$

$$x^2 = 16$$

$$x = \sqrt{16}$$

$$x = 4$$

(c)

$$10^2 = x^2 + 8^2$$

$$x^2 = 100 - 64$$

$$x^2 = 36$$

$$x = \sqrt{36}$$

$$x = 6$$

Pages 50-52

A. $\overline{ZY}$

B. $\overline{IJ}$, $\overline{GJ}$

ie: $\overline{HI} \cong \overline{GJ}$ and $\overline{HG} \cong \overline{IJ}$

$\angle G \cong \angle I$ and $\angle H \cong \angle J$

C. $\overline{NM}$, $\overline{LM}$, 3.8 cm

$\angle K \cong \angle M$, $\angle N \cong \angle L$

D. $\overline{DC}$, $\overline{AD}$, 5.1 cm , 4.4 cm 90°

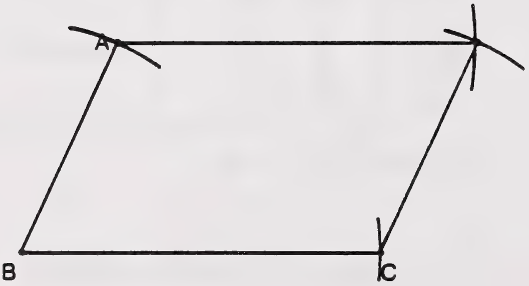
E. $\overline{PQ}$, $\overline{PS}$, 3.1 cm 90°

Pages 53-55

Exercise – Quadrilaterals

1. (a) true (b) false (c) true (d) true
(e) false (f) false (g) true (h) false
(i) false (j) true

2.



3. (a) square (b) trapezoid (c) parallelogram (d) rectangle
(e) rhombus (f) square (g) rhombus (h) square
(i) rectangle (j) parallelogram

	Parallelogram	Rectangle	Rhombus	Square
1.	yes	yes	yes	yes
2.	no	yes	no	yes
3.	no	no	yes	yes
4.	no	no	yes	yes

Page 56

1. $= |-3 + 8| = |5| = 5$
2. $|6 - 14| = |-8| = 8$
3. $= |-5 - 9| = |-14| = 14$
4. $|2 - (-7)| = |2 + 7| = |9| = 9$

Pages 67-68

Exercise – Coordinate Geometry

1. (a) neither (b) parallel (c) neither
   (d) perpendicular (e) perpendicular (f) parallel

2. $3x + 7y - 21 = 0$

$$7y = -3x + 21$$

$$\text{slope} = \frac{-3}{7}$$

$$14x - 6y + 28 = 0$$

$$-6y = -14x - 28$$

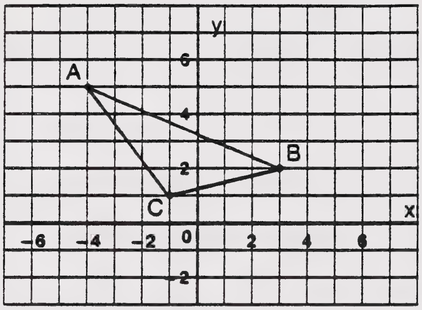
$$y = \frac{-14x}{-6} + \frac{-28}{-6}$$

$$y = \frac{7}{3}x + \frac{14}{3}$$

$$\text{slope} = \frac{7}{3}$$

Since the slopes are negative reciprocals of each other, the lines are perpendicular.

3.



(a) $AB = \sqrt{7^2 + (-3)^2} = \sqrt{49 + 9} = \sqrt{58} = 7.616$

$AC = \sqrt{(-4 + 1)^2 + (5 - 1)^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

$BC = \sqrt{(3 + 1)^2 + (2 - 1)^2} = \sqrt{4^2 + 1^2} = \sqrt{16 + 1} = \sqrt{17} = 4.123$

(b) Midpoint of $\overline{AB} = \left(-\frac{1}{2}, \frac{7}{2}\right)$

Midpoint of $\overline{AC} = \left(\frac{-4 + -1}{2}, \frac{5 + 1}{2}\right) = \left(\frac{-5}{2}, \frac{6}{2}\right) = \left(\frac{-5}{2}, 3\right)$

Midpoint of $\overline{BC} = \left(\frac{-1 + 3}{2}, \frac{1 + 2}{2}\right) = \left(\frac{2}{2}, \frac{3}{2}\right) = \left(1, \frac{3}{2}\right)$

(c) Slope of $\overline{AB} = \frac{-3}{7}$

Slope of $\overline{AC} = \frac{1 - 5}{-1 + 4} = \frac{-4}{3}$

Slope of $\overline{BC} = \frac{2 - 1}{3 + 1} = \frac{1}{4}$

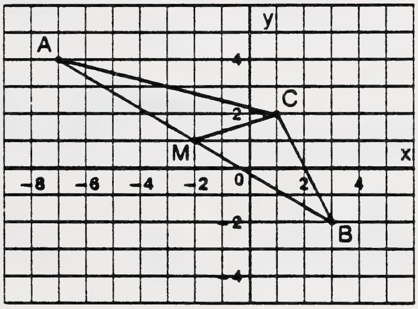
4. Slope of $\overleftrightarrow{BC} = \dfrac{0 - 5}{-4 - 2} = \dfrac{-5}{-6} = \dfrac{5}{6}$

Slope of any line parallel to $\overleftrightarrow{BC}$ is also $\dfrac{5}{6}$.

Equation of line on A(2, −3) with a slope of $\dfrac{5}{6}$ is:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-3) &= \frac{5}{6}(x - 2) \\ 6y + 18 &= 5x - 10 \\ 6y &= 5x - 28 \\ y &= \frac{5}{6}x - \frac{28}{6} \\ y &= \frac{5}{6}x - \frac{14}{3} \end{aligned}$$

5.



Midpoint of $\overline{AB} = \left(\dfrac{-7 + 3}{2}, \dfrac{4 - 2}{2}\right)$

$= \left(\dfrac{-4}{2}, \dfrac{2}{2}\right)$

$= (-2, 1)$

C = (1,2) , M = (−2,1)

$\overline{CM} = \sqrt{(1 + 2)^2 + (2 - 1)^2}$

$= \sqrt{3^2 + 1^2}$

$= \sqrt{10}$

$= 3.162$

6. $5x - 2y + 12 = 0$

$$-2y = -5x - 12$$

$$y = \frac{5}{2}x + 6$$

$\therefore$ Slope $= \frac{5}{2}$.

A line $\perp$ to this line has a slope of $\dfrac{-1}{\frac{5}{2}} = \dfrac{-2}{5}$.

The equation of a line on $(6, -2)$ with a slope of $\frac{-2}{5}$ is: $y - y_1 = m(x - x_1)$

$$\begin{aligned} y - (-2) &= \frac{-2}{5}(x - 6) \\ 5y + 10 &= -2x + 12 \\ 5y &= -2x + 2 \\ y &= \frac{-2}{5}x + \frac{2}{5} \end{aligned}$$

7. (a) $AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2 - 3)^2 + (2 - 9)^2} = \sqrt{(-1)^2 + (-7)^2}$

$$= \sqrt{1 + 49} = \sqrt{50}$$

$BC = \sqrt{(8 - 3)^2 + (4 - 9)^2} = \sqrt{5^2 + 5^2} = \sqrt{25 + 25} = \sqrt{50}$

(b) $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2 + 8}{2}, \frac{2 + 4}{2}\right) = \left(\frac{10}{2}, \frac{6}{2}\right) = (5,3)$

(c) Slope of $\overline{CM} = \frac{9 - 3}{3 - 5} = \frac{6}{-2} = -3$

Slope of $\overline{AB} = \frac{2 - 4}{2 - 8} = \frac{-2}{-6} = \frac{1}{3}$

Since the slopes are negative reciprocals, $\overline{CM} \perp \overline{AB}$.

END OF LESSON J

Lesson K

Page 4

C.

Prefix	Symbol	Meaning in Words	Meaning in Numbers	Power of 10
mega	M	million	1 000 000	10^6
kilo	k	thousand	1 000	10^3
hecto	h	hundred	100	10^2
deca	da	ten	10	10^1
deci	d	tenth	0.1	10^{-1}
centi	c	hundredth	0.01	10^{-2}
milli	m	thousandth	0.001	10^{-3}
micro	μ	millionth	0.000 001	10^{-6}

Pages 10-11 **Exercise – Metric System**

1. (a) kg (b) L (c) kL
(d) 10 (e) 100 (f) dL
(g) L (h) 0.005 (i) 10 000
(j) 0.08 (k) 0.1 (l) 2550
(m) 160 (n) 150 (o) 30
(p) 19 000 (q) 800 (r) 670

2. (a) kilogram (b) mL (c) centimeter
(d) kL (e) L

3. (a) 0.01 (b) 100 (c) 0.001 (d) 1000
(e) 0.1 (f) 10 (g) 1000 (h) 0.001
(i) 1000 (j) 0.001 (k) 0.001 (l) 0.001
(m) 0.001 (n) 1000 (o) 1000 (p) 1000

4. (a) 26 cm (b) 51 mm

5. (a)

square millimetres: **mm^2**

square decimetres: **dm^2**

(b)

square decametres: **dam^2**

square hectometres: **hm^2**

Page 12

1 cm = 10 mm , so 1 cm^2 = 10 mm × 10 mm = 100 mm^2

1 dm = 10 cm , so 1 dm^2 = 10 cm × 10 cm = 100 cm^2

Page 15

cubic decimetre: **dm^3**

cubic kilometre: **km^3**

cubic hectometre: **hm^3**

cubic decametre: **dam^3**

Pages 20-21 **Exercise – Area, Volume, Mass and Capacity**

1. 7.2 L = **7.2** dm^3

632 g = **0.632** kg

2 hm^2 = **20 000** m^2

121 cm^3 = **121** mL

75 hm^2 = **75** ha

35 kg = **35 000** g

50 000 mg = **50** g

83.7 g = **83 700** mg

10 000 g = **10** kg

2 550 mg = **2.55** kg

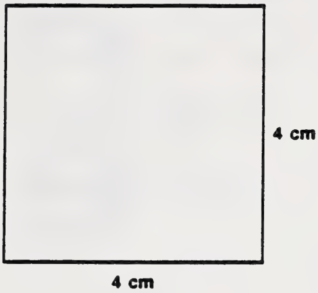
23.6 g = **323 600** mg

10 dm^3 = **10** L

920 kg = **920 000** g

15 ha = **150 000** m^2

2. (a)



Length = 4 cm

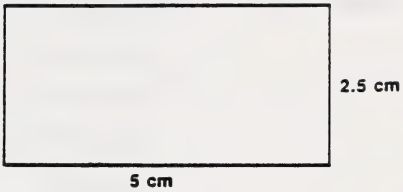
Width = 4 cm

Area = Length × Width

= 4 cm × 4 cm

= **16 cm^2**

(b)



Length = 5 cm

Width = 2.5 cm

Area = Length × Width

= 5 cm × 2.5 cm

= **12.5 cm^2**

3. $= \frac{9720\ cm^2}{81\ cm} = 120\ cm$ $= \frac{2444\ mm^2}{47\ mm} = 52\ mm$

4. Volume = 2 m × 0.7 m × 5 m

= 7 m^3

The capacity of the tank is **7000 L** or **7 kL.**

5. Volume = 3.2 m × 3.2 m × 3.2 m

= 32.768 m^3

The mass of the water is **32.768 t** or **32 768 kg**.

Page 26

$$\sin \angle M = \frac{9}{15}$$

$$\cos \angle M = \frac{\text{side adjacent to } \angle M}{\text{hypotenuse}} = \frac{MO}{MN} = \frac{12}{15}$$

$$\tan \angle M = \frac{\text{side opposite to } \angle M}{\text{side adjacent to } \angle M} = \frac{NO}{MO} = \frac{9}{12}$$

Pages 27-28 **Exercise – Primary Trigonometric Ratios**

1. (a) $\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$

(b) $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{5}}{3}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{2}{3}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{5}}{2}$

(c) $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{12}{13}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{13}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{12}{5}$

2. (a) $c^2 = 225 - 81$

$c^2 = 144$

$c = 12$

$\sin C = \frac{\text{opp}}{\text{hyp}} = \frac{12}{15}$

$\cos C = \frac{\text{adj}}{\text{hyp}} = \frac{9}{15}$

$\tan C = \frac{\text{opp}}{\text{adj}} = \frac{12}{9}$

(b) $2^2 = b^2 + 1^2$

$4 = b^2 + 1$

$b^2 = 4 - 1$

$b^2 = 3$

$b = \sqrt{3}$

$\sin B = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$

$\cos B = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$

$\tan B = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{1}$

Page 29 $\cos A = \frac{4}{4.9} = 0.82$ $\tan A = \frac{2.8}{4} = 0.70$

Pages 32-33 **Exercise – Trigonometric Ratios**

1. (a) 0.0349 (b) 0.5000 (c) 0.8660 (d) 1.000
(e) 0.5000 (f) 0.9336 (g) 0.6561 (h) 0.9903

2. (a) 45° (b) 89° (c) 0° (d) 45°
(e) 69° (f) 40° (g) 76° (h) 24°

3. (a) 0.0000 to 1.0000
(b) 1.0000 to 0.0000
(c) increases from 0.0000 to 57.29

4. (a) (ii) sin A
(iii) cos A

(b) (i) cos B
(ii) tan B
(iii) sin B

(c) (i) cos X
(ii) tan X
(iii) sin X

(d) (i) tan Z
(ii) cos Z
(iii) sin Z

Page 35

$$\cos 63^\circ = \frac{CB}{1.65}$$

$$CB = 1.65 \times \cos 63^\circ$$

$$= 1.65 \times 0.4540$$

$$= 0.749\ 10$$

$\therefore$ CB = 0.75 (to 2 decimal places)

Pages 37-38 **Exercise – Solving Right Triangles**

1. $\angle B = 90 - 67 = \mathbf{23^\circ}$

AB: $\sin 67^\circ = \dfrac{90^\circ}{AB}$

$$AB = \frac{90}{\sin 67^\circ}$$

$$AB = \frac{90}{0.9205}$$

$$= \mathbf{97.8}$$

AC: $\tan 67^\circ = \dfrac{90^\circ}{AC}$

$$AC = \frac{90}{\tan 67^\circ}$$

$$= \frac{90}{2.3559}$$

$$= \mathbf{38.2}$$

2. $\sin P = \dfrac{340}{420} = 0.809\ 52$

$\therefore \angle P = \mathbf{54^\circ}$

$\cos R = \dfrac{340}{420} = 0.809\ 52$

$\therefore \angle R = \mathbf{36^\circ}$

$$PQ^2 = 420^2 - 340^2$$

$$= 176\ 400 - 115\ 600$$

$$= 60\ 800$$

$$PQ = 246.57$$

$\therefore$ **PQ = 247**

3. $\angle O = 90° - 20° = \mathbf{70°}$

MN: $\cos 20° = \dfrac{MN}{25}$

$MN = 25 \cos 20°$

$= 23.49$

$MN = \mathbf{23}$

MO: $\sin 20° = \dfrac{MO}{25}$

$MO = 25 \sin 20°$

$= 8.55$

$MO = \mathbf{8.6}$

4. $\angle A$: $\tan A = \dfrac{35}{27} = 1.2963$

$\therefore \angle \mathbf{A = 52°}$

$\angle B$: $\tan B = \dfrac{27}{35} = 0.7714$

$\therefore \angle \mathbf{B = 38°}$

AB: $AB^2 = 35^2 + 27^2$

$= 1225 + 729$

$= 1954$

$\therefore \mathbf{AB = 44.2}$

Page 46 **Exercise – Solving Problems with Trigonometry**

1. $2 \text{ min} = \dfrac{2}{60} = 0.0\dot{3}$ hours

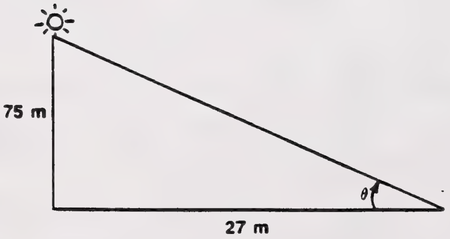
$d = r \times t = 10 \times 0.0\dot{3} = 0.\dot{3}$

$\tan 38° = \dfrac{x}{0.333}$

$x = 0.7813 \times 0.333 = 0.260$

The ship is **0.26 km** from the lighthouse.

2.



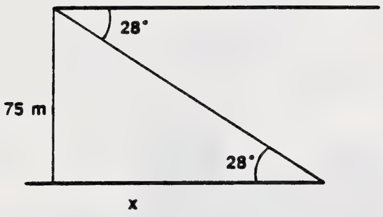
$\tan \theta = \dfrac{15}{27} = 0.5556$

$\tan \theta = 0.5556$

$\theta = 29°$

The angle of elevation of the sun is **29°**.

3.



$$\tan 28^\circ = \frac{75}{x}$$

$$x = \frac{75}{\tan 28^\circ}$$

$$x = \frac{75}{0.5317}$$

$$x = 141 \text{ m}$$

The boat is **141 m** from the base of the cliff.

4. $$\cos 32^\circ = \frac{6.2}{d}$$

$$d = \frac{6.2}{\cos 32^\circ}$$

$$d = \frac{6.2}{0.8480}$$

$$d = 7.3109$$

The missile is **7.31 km** from the observer.

Page 52

Exercise – Using Tables to Find Reciprocal Trigonometric Values

1. $\sin \theta = \frac{3}{5}$ **—opposite** **—hypotenuse**

$(\text{adj})^2 = 5^2 - 3^2 = 25 - 9 = 16$

$\therefore$ adj side $= 4$

Then $\sin \theta = \frac{3}{5}$ $\csc \theta = \frac{5}{3}$

$\cos \theta = \frac{4}{5}$ $\sec \theta = \frac{5}{4}$

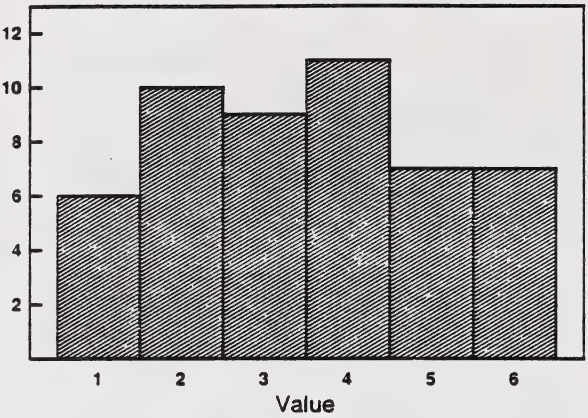
$\tan \theta = \frac{3}{4}$ $\cot \theta = \frac{4}{3}$

2. (a) 1.0864 (b) 0.1051 (c) 11.474 (d) 2.1310

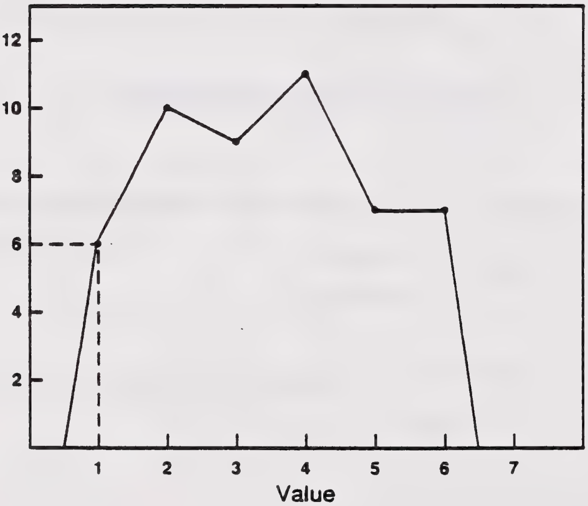
Page 56

1.

Histogram



Frequency Polygon



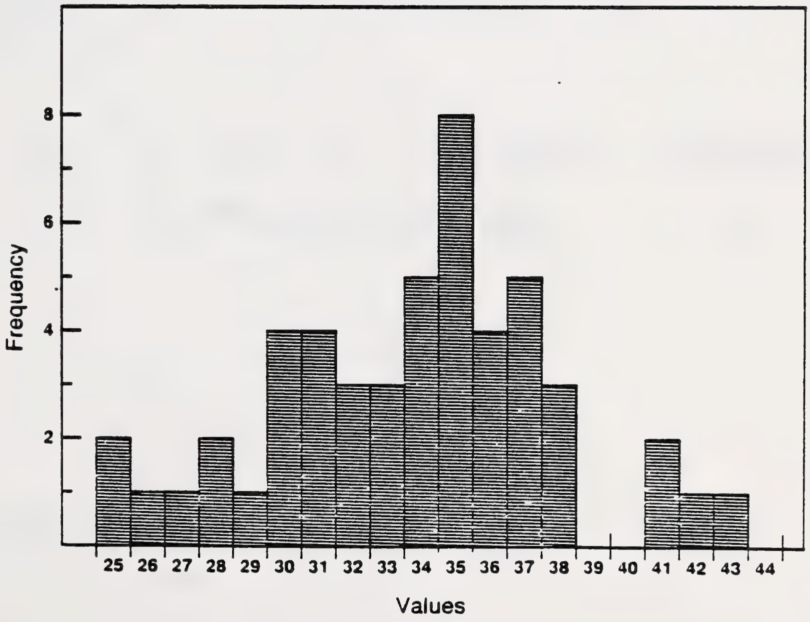
2. (a)

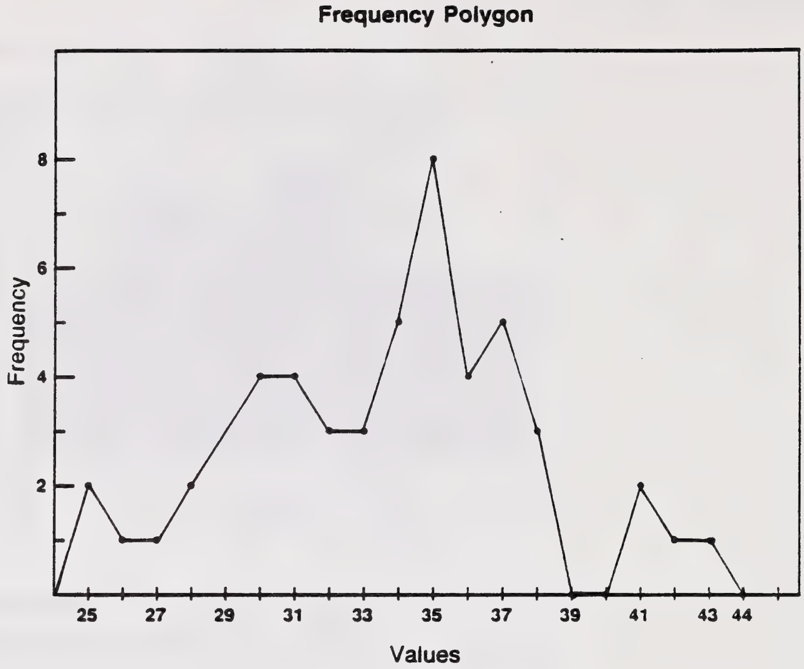
Frequency Distribution Table

Value	Tally	Frequency
25	||	2
26	|	1
27	|	1
28	||	2
29	|	1
30	||||	4
31	||||	4
32	|||	3
33	|||	3
34	𝍸	5

Value	Tally	Frequency
35	𝍸 |||	8
36	||||	4
37	𝍸	5
38	|||	3
39		0
40		0
41	||	2
42	|	1
43	|	1
44		0

Histogram





Pages 59-60 **Exercise**

1. $$\text{Mean} = \frac{105 + 86 + 92 + 120 + 75}{5}$$

$$= \frac{478}{5}$$

$$= 95.6 \text{ m}$$

The mean gain per game is **95.6 m**.

2. $$\text{Mean} = \frac{5.8 + 6.1 + 5.5 + 6.3 + 5.9 + 5.2 + 6.4 + 5.7 + 6.1}{9}$$

$$= \frac{53}{9}$$

$$= 5.9$$

The mean wrist size is **5.9 cm**.

Pages 65-68 **Exercise – Measures of Central Tendency**

1. (a) **14 500 , 14 800 , 15 300 , 15 300 , 15 500 , 16 000 , 38 750**

The mode is **15 300**.

(b) **15 300**

(c) $$\text{Mean} = \frac{14500 + 14800 + 15300 + 15300 + 15500 + 16000 + 38750}{7}$$

$$= \frac{130150}{7}$$

$$= \$18\ 593$$

The mean salary is **$18 593**.

No.

Why? Kelly's salary is an extreme, making the mean appear to be much higher.

median or mode

2. (a) mean (b) mean or median (c) mode (d) mean

(e) median (f) mode (g) mean (h) median

3. (a) 9 , 10 , 13 , 15 , 15 , 15 , 16 , 19 , 20

The mode is 15.

The median is 15.

$$\text{The mean} = \frac{9 + 10 + 13 + 15 + 15 + 15 + 16 + 19 + 20}{9}$$

$$= 14.67$$

$$= 15$$

The mean number of accidents is **15**.

(b) mean

4. Median for A is 15.

$$\text{Mean} = \frac{12 + 14 + 15 + 18 + 20}{5}$$

$$= \mathbf{15.8}$$

Median for B is 16.5.

$$\text{Mean} = \frac{12 + 14 + 15 + 18 + 20 + 92}{6}$$

$$= \mathbf{28.5}$$

... 92 is an extreme value in B.

... 92 is an extreme value.

median

5. (a) 6, 8, 9, 10, 10, 12, 12, 13, 14

The median is 10.

The mode is 10 and 12.

$$\text{The mean} = \frac{6 + 8 + 9 + 10 + 10 + 12 + 12 + 13 + 14}{9} = 10.4$$

(b) 52, 58, 60, 62, 64, 68, 69, 73

$$\text{The median is } \frac{62 + 64}{2} = 63.$$

The mode is none.

$$\text{The mean} = \frac{52 + 58 + 60 + 62 + 64 + 68 + 69 + 73}{11} = 63.3.$$

(c) 0, 3, 4, 5, 6, 6, 7, 8, 9, 9, 12

The median is 6.

The mode is 6 and 9.

$$\text{The mean} = \frac{0 + 3 + 4 + 5 + 6 + 6 + 7 + 8 + 9 + 9 + 12}{11} = 6.3$$

Pages 72-73

1. (a) Range = 24 − 3 = 21

   (b) Range = 48 − 5 = 43

   (c) Range = 61 − 8 = 53

2. (a) $$\text{Mean} = \frac{6 + 8 + 8 + 9 + 9 + 9 + 10 + 10 + 12}{9}$$

$$= \frac{81}{9}$$

$$= 9$$

Value	Differences	Squares
x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
6	6 − 9 = −3	9
8	8 − 9 = −1	1
8	8 − 9 = −1	1
9	9 − 9 = 0	0
9	9 − 9 = 0	0
9	9 − 9 = 0	0
10	10 − 9 = 1	1
10	10 − 9 = 1	1
12	12 − 9 = 3	9

$$\text{The mean of the squares} = \frac{9 + 1 + 1 + 1 + 1 + 9}{9}$$

$$= \frac{22}{9}$$

$$= 2.44$$

$$\sigma = \sqrt{2.44}$$

σ = 1.6 (rounded to one decimal place)

(b) $$\text{Mean} = \frac{20 + 24 + 17 + 15 + 28 + 25 + 22 + 20 + 17 + 12 + 20}{11}$$

$$= \frac{220}{11}$$

$$= 20$$

Value x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
12	$12 - 20 = -8$	64
15	$15 - 20 = -5$	25
17	$17 - 20 = -3$	9
17	$17 - 20 = -3$	9
20	$20 - 20 = 0$	0
20	$20 - 20 = 0$	0
20	$20 - 20 = 0$	0
22	$22 - 20 = 2$	4
24	$24 - 20 = 4$	16
25	$25 - 20 = 5$	25
28	$28 - 20 = 8$	64

The sum of the squares $= 64 + 25 + 9 + 9 + 4 + 16 + 25 + 64 = 216$

The mean of the squares $= \frac{216}{11}$

$$= 19.6$$

$$\sigma = \sqrt{19.6}$$

$$\sigma = 4.4$$

END OF LESSON K